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ON THE PROPAGATION, REFLEXION, TRANSMISSION AND STABILITY OF ATMOSPHERIC ROSSBY-GRAVITY WAVES ON A BETA-PLANE IN THE PRESENCE OF LATITUDINALLY SHEARED ZONAL FLOWS

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CONTENTS

The propagation properties of Rossby-gravity waves in an isothermal atmosphere on a beta-plane are investigated in the presence of a latitudinally sheared zonal flow. The perturbation equation is found to possess *seven* regular singularities provided the fluid is *non*-Boussinesq, and only *five* for Boussinesq fluids. In slowly varying shear a local dispersion relation is derived and used to study the wave normal surfaces and ray trajectories. The cross sections of the wave normal surfaces in horizontal planes possess three critical latitudes occurring where the intrinsic frequency $\hat{\omega}$ takes the values 0,

$\pm N$, where N is the Brunt–Väisällä frequency. The former is the usual Rossby wave critical latitude (R.w.c.l.) and the latter are essentially gravity wave critical latitudes (g.w.c.l.). Waves can propagate only on one side of a R.w.c.l. while propagation is possible on both sides of a g.w.c.l. provided the vertical wavenumber, m, there is *real* and non-zero. Also for real values of m and provided the atmosphere is *non*-Boussinesq the g.w.c.l. exhibits valve-like behaviour. Such valve behaviour is shown to be responsible for aiding high frequency waves (i.e. gravity waves) to penetrate jet-like wind streams and may facilitate the transfer of energy and momentum across latitudes.

The full wave treatment shows that the system possesses a wave-invariant which has a simple physical interpretation only when m is real in which case it represents the conservation of the total northward wave energy flux. The invariant is used, together with the legitimate solutions near the critical latitudes, to study the influence of each of the critical latitudes on the intensity of the wave. It is found that the R.w.c.l. can be associated with energy absorption or emission, depending on certain specified conditions, but the g.w.c.l. is always associated with energy absorption although the amount of energy absorbed depends crucially on whether m is real or imaginary.

The reflexion, transmission and stability of atmospheric waves by a finite shear, thickness L, are also studied, by using a full wave treatment in the presence of general flow profiles. For smoothly varying shear flows the use of the wave-invariant yields a relation between the reflexion and transmission coefficients. It is then deduced that in the absence of critical latitudes within the shear over-reflexion (i.e. the amplitude of the reflected wave exceeds that of the incident one) is possible only for real values of m in which case planetary waves incident on the shear are transmitted as gravity waves on the far side of the shear. Such over-reflecting régimes are, however, due to the presence of natural modes of the system. Moreover, it is found possible to isolate certain situations in which the R.w.c.l. within the shear *enhances* over-reflexion.

The situation when the shear is linear and thin is studied analytically. Explicit expressions for the reflexion and transmission coefficients are obtained. It is then shown that over-reflexion is present in a stable shear in which case the energy is extracted from the mean flow primarily at the R.w.c.l. The influence of a general flow on the R.w.c.l. is also studied in a special case to show that the reflectivity and stability properties of the thin shear are strongly dependent on the type of shear present. Comparison of these results with those obtained for the corresponding vortex sheet show wide disagreement. A general criterion for determining whether a vortex sheet will adequately represent the corresponding thin shear layer is offered in the final section.

1. Introduction

Rossby (1939), in his study of the relation between variations in the intensity of the zonal circulation of the atmosphere and the displacements of the semi-permanent centres of action, approximated the spatial variations of the vertical component, f, of the angular velocity of the earth in the neighbourhood of a point latitude θ_0, by

$$f = f_0 + \beta y,$$

where y is the northward distance measured from the point and

$$f_0 = 2\Omega \sin\theta_0, \quad \beta = (2\Omega/a)\cos\theta_0,$$

where Ω is the magnitude of the angular velocity and a is the mean radius of the Earth. Rossby also neglected the horizontal component of the Earth's rotation. This approximation is now known as the β-plane approximation and the waves whose existence is due to this β-effect are generally known as Rossby (or planetary) waves. The main advantage of this approximation is to permit the neglect of the curvature of the Earth except in as much as it affects the magnitude of the vertical component of $\mathbf{\Omega}$.

Longuet-Higgins (1964*a*, 1965) studied the propagation of planetary waves both when the plane is infinite and bounded and compared the results with those obtained by direct methods. One of the important results emerging from these studies is that the β-plane approximation provides an adequate representation of the local behaviour of planetary (Rossby) waves in thin spherical shells (like the Earth's atmosphere) except in the neighbourhoods of two 'singular' latitudes (one on either side of the equator). This result has not only added support to the plausibility of conclusions about atmospheric waves reached using a β-plane approximation (Rossby 1939; Charney & Drazin 1961; Drazin & Howard 1966) but has also initiated further studies on various aspects of wave motions on a beta-plane. Longuet-Higgins (1964*b*) in his study on the group velocity and energy flux of planetary waves found that planetary waves (in the absence of a flow) can only propagate (in phase) westward but both westward and eastward energy propagation (as defined by the direction of the group velocity) are possible. Lighthill (1967*a*) studied the generation of Rossby waves by travelling forcing effects. McKee (1972) examined the scattering of Rossby waves by partial barriers in an attempt to estimate the energy penetrating the Drake Passage between South America and Antarctica. Studies on the stability of Rossby waves in the absence of a shear flow were made by a number of authors (see, for example, Gill 1974; Coaker 1977) and for particular basic flows by, for example, Dickinson & Clare (1973); Geisler & Dickinson (1974).

The beta-plane approximation has also been used in magnetohydrodynamics to obtain simple solutions amenable to exact treatment in order to understand the basic physical processes taking place in complex systems. Hide (1966) in his study of the geomagnetic secular variations initiated such an approach by investigating the propagation of hydromagnetic planetary waves in thin spherical shells to obtain a simple dispersion relation which was later studied extensively by Hide & Jones (1972). The propagation of hydromagnetic planetary waves through latitudinal velocity and magnetic shear, as well as their reflexion by shear layers, have recently been studied by Eltayeb & McKenzie (1977).

The propagation of planetary waves in zonal flows which vary with altitude was studied by Charney & Drazin (1961) in their investigation of the energy transfer from the lower to the upper atmosphere. They found that, with parameters appropriate to atmospheric conditions, planetary waves are evanescent in the vertical direction and concluded that little energy transfer can take place by planetary waves. Dickinson (1968) examined the propagation of planetary waves sheared latitudinally and pointed out the existence of critical latitudes, occurring where the intrinsic frequency $\hat{\omega} = 0$, where the northward transfer of zonal momentum experiences a finite jump.

Recently the propagation of planetary waves on a beta-plane in the presence of latitudinally sheared zonal flows has been studied by Mekki & McKenzie (1977, hereinafter referred to as M.M.). In M.M. the general wave equation for an isothermal atmosphere was derived but a detailed study was made when the assumptions

(i) $\hat{\omega}^2 < f^2$,
(ii) $\hat{\omega}^2 < N^2$,
(iii) Boussinesq approximation

were applicable. Here $\hat{\omega}$ is the intrinsic frequency defined by $\hat{\omega} = \omega - kU$, where ω is the frequency, k the zonal wavenumber and U the zonal flow; N being the Brunt–Väisällä (buoyancy) frequency. The propagation properties in different types of zonal flow were studied and the various types of ray trajectory were identified. The reflexion of the waves by a vortex sheet was investigated and the solutions near the critical latitudes were examined.

Although the study in M.M. is comprehensive, a number of questions on the general propagation properties of wave motions (in latitudinally sheared zonal flows) in the atmosphere remain unanswered. In view of the fact that gravity waves (frequency $\omega \approx N$) are known to exist in the atmosphere (Hough 1898) and believed to account for some of the energy transfer to the ionosphere (Hines 1959; Hines & Reddy 1967) in addition to their importance in the planetary boundary layer (see, for example, Davis & Peltier 1976), what are the circumstances, if any, in which a planetary wave ($\omega \ll N$) can transform into a gravity wave, if we take into account that although ω may be small compared to N, the intrinsic frequency $\hat{\omega}$ may compare with N if the flow speed is large enough? What is the influence of the vertical wavenumber on the latitudinal transfer of energy and momentum? What is the influence of the critical latitudes on the reflectivity, transmissivity and stability of atmospheric waves riding a general profile of a basic flow? In the present study we attempt to shed some light on the answers to these questions, by extending the study in M.M. to the régime in which the restrictions (i)–(iii) above are absent.

In §2 we briefly derive the full wave equation governing perturbations in an isothermal compressible atmosphere in the presence of latitudinally sheared zonal flow. In the absence of the restrictions (i)–(iii) made in M.M. the wave equation possesses *seven* singularities as compared to only three predicted by the M.M. theory. An examination of the influence of compressibility, Boussinesq approximation and low frequency restriction on the system shows that the effect of compressibility is mainly to introduce acoustic-gravity waves when the flow speeds are comparable with the speed of sound. Realizing that most wind speeds in the atmosphere are subacoustic the analysis is restricted to incompressible fluids. However, the influence of the Boussinesq approximation is found to have rather drastic effects on the system: it not only reduces the singularities of the system to five but it also annuls a valve behaviour which, in general, occurs at the critical latitudes $\hat{\omega} = \pm N$, when the vertical wavenumber, m, is *real*. Incidentally the nature of m (i.e. real or imaginary) has a very strong influence on the type of solutions that can prevail in the vicinities of the critical latitudes $\hat{\omega} = \pm N$.

In §3 we study the propagation of Rossby-gravity waves in slowly varying shear flows. When the wavelength of the waves is much smaller than the scale of variations of the basic flow, a W.K.B.J. solution yields a *local* dispersion relation. This relation is examined geometrically for all frequencies $\omega < N$. The cross-sections of the wave normal surfaces in the (k, l) plane, where l is the local northward component of the wave vector, showed that three critical latitudes, occurring where $\hat{\omega} = 0$, $\pm N$, are possible. The critical latitude $\hat{\omega} = 0$ is the usual planetary (Rossby) wave critical latitude (R.w.c.l.) while the critical latitudes at $\hat{\omega} = \pm N$ resemble gravity wave critical latitudes (g.w.c.l.), which have been the subject of study by a number of authors (see, for example, Booker & Bretherton 1967; Baldwin & Roberts 1970; Breeding 1971; Eltayeb & McKenzie 1975), although a detailed analysis of the present g.w.c.l. showed that its influence on the intensity of the wave is essentially different from that of the classical gravity wave critical level because of the presence of rotation. This becomes particularly evident when the reflexion of atmospheric waves by a finite shear is investigated in §6.

The wave normal curves in the (k, l) plane show that planetary waves propagating in a flow increasing with latitude are reflected towards their critical latitudes but at higher flow speeds gravity waves appear, the regions of propagation of the two waves, being separated by a domain of evanescence. This state of affairs is pertinent only to the slowly varying basic state. In a full wave treatment the region of evanescence is modified considerably and as a consequence planetary waves propagating into strengthening wind profiles are transformed into gravity waves (in the sense that energy and momentum is transferred from the planetary wave to the gravity wave through the interaction with the mean flow). Such situations are shown in § 6 to give rise to over-reflexion and instability.

Propagation in the vicinities of the g.w.c.ls is found to depend crucially on m. If m is real, propagation is possible on *both* sides of the g.w.c.l. A wave approaching such a critical latitude will either be absorbed or transmitted according to whether

$$v_G \, mfglU\hat{\omega} \gtrless 0,$$

at the critical latitude, where v_G is the local group velocity component in the northward direction and g the gravitational acceleration. If m is imaginary no propagation occurs near the g.w.c.ls (although a full wave treatment indicates that the solutions are oscillatory on both sides but no valve behaviour is present). In the case of two-dimensional wave motion (i.e. $m = 0$) propagation is possible only on one side and a full wave treatment shows that the situation is identical to that of the classical gravity wave critical level when the Richardson number is $\frac{1}{4}$.

Adopting the group velocity approach (Lighthill 1965) the local dispersion relation is used to construct the various ray trajectories that can arise in different flow profiles. In jet-like streams planetary waves can propagate across the jet if the maximum speed, U_m, is small, but as U_m increases planetary waves are 'pushed' to the wings of the jet and gravity waves appear around the centre. Despite the existence of g.w.c.ls there, gravity waves can, in certain circumstances and by virtue of the valve behaviour, propagate right across the centre of the jet even in the presence of *two* g.w.c.ls (cf. § 3).

In § 4 we derive the wave-invariant, $\mathscr{A}$, of the system. Here m again plays an important role. If m is real then $\mathscr{A}$ is proportional to the total wave energy flux, F, in the northward direction but for imaginary m, F is *not* conserved. Section 5 deals with the detailed analysis for the solutions in the neighbourhoods of the singularities of the system.

In § 6 we investigate the reflexion by and transmission through a shear layer, thickness L. For arbitrary values of L only a general relation between the reflexion and transmission coefficients is derived by using the wave-invariant $\mathscr{A}$. This relation is, however, sufficient to show the possibility of over-reflecting régimes even in the absence of critical latitudes within the shear. It is also possible to isolate certain circumstances in which the *presence* of a critical latitude *enhances* over-refiexion. These general results are confirmed by a detailed analysis for the case of a thin shear (in the sense that $kL \ll 1$). Here explicit expressions for the reflexion and transmission coefficients are obtained and the stability of the thin shear is also examined. The results show the existence of stable over-reflecting régimes; a result which has already been shown to apply for the current-vortex sheet in some hydromagnetic systems (Acheson 1976; Eltayeb & McKenzie 1977). However, a comparison of the thin shear layer results with those obtained for the corresponding vortex sheet treatment in M.M. shows that agreement is not satisfactory. The reasons for the discrepancy in the vortex sheet treatment in the present case are mainly due to the presence of critical latitudes within the shear. By considering a variety of other situations it seems possible to identify the cases in which sudden discontinuities would provide a reasonable representation of the thin shear layer model (see § 7).

2. The wave equation

The equations governing the propagation of atmospheric Rossby-gravity waves on a beta-plane in the presence of a zonal flow sheared latitudinally have been derived in M.M. Unfortunately, however, their wave equation is incorrect although most of the general conclusions they reached, within the context of the assumptions they made, still hold good. For the purpose of the study presented below it is necessary to obtain the correct wave equation. This will be particularly evident when we discuss the wave normal surfaces in § 3 and the critical latitudes of the system (see §§ 5 and 6).

The basic equations governing motions in an isothermal compressible dissipationless rotating fluid are those of momentum, continuity and energy. They are respectively

$$\rho(\mathrm{D}\boldsymbol{u}/\mathrm{D}t+2\boldsymbol{\Omega}\wedge\boldsymbol{u}) = -\nabla p+\rho\boldsymbol{g}, \tag{2.1}$$

$$\mathrm{D}\rho/Dt+\rho\nabla\cdot\boldsymbol{u} = 0, \tag{2.2}$$

$$\mathrm{D}p/\mathrm{D}t-c^2\mathrm{D}\rho/\mathrm{D}t = 0, \tag{2.3}$$

in which the mobile operator, $\mathrm{D}/\mathrm{D}t$, is defined by

$$\mathrm{D}/\mathrm{D}t\equiv\partial/\partial t+\boldsymbol{u}\cdot\nabla. \tag{2.4}$$

Here ρ is the density, $\boldsymbol{u}$ the velocity, $\boldsymbol{\Omega}$ the angular velocity, p the pressure, g the gravitational acceleration, t the time, c the speed of sound (assumed uniform) and ∇ is the standard gradient operator.

We consider a cartesian system of coordinates Oxyz at the locality of the point O such that Ox, Oy and Oz lie along the east, north and upward vertical respectively. We take a basic state in which the velocity, pressure, density and angular velocity assume the values $\boldsymbol{u}_0$, p_0, ρ_0 and $\frac{1}{2}f\hat{\boldsymbol{z}}$, where $\hat{\boldsymbol{z}}$ is a unit vector in the direction of z increasing, and

$$\boldsymbol{u}_0 = U(y)\,\hat{\boldsymbol{x}},\quad f=f_0+\beta y,\quad \frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z} = -\frac{1}{H} = \text{const.}, \tag{2.5}$$

in which $\hat{\boldsymbol{x}}$ is a unit vector along Ox. The basic state pressure is then governed by

$$\partial p_0/\partial y = -\rho_0 fU,\quad \partial p_0/\partial z = -\rho_0 g, \tag{2.6}$$

which yield an equation for the variation of ρ_0 with latitude

$$\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial y} = -fU/gH = -a(y),\quad \text{say}. \tag{2.7}$$

In (2.5) and (2.7) $\quad f_0 = 2\Omega\sin\theta_0,\quad \beta = (2\Omega/a)\cos\theta_0,\quad H = \bar{R}T/g \qquad (2.8)$

in which θ_0 is the latitude of O, T the uniform temperature of the atmosphere, $\bar{R}$ is the universal gas constant and a the mean radius of the Earth. It may be remarked here that the neglect of the horizontal component of $\boldsymbol{\Omega}$ is physically plausible provided θ_0 is not too small (Longuet-Higgins 1965).

We now perturb the basic state by letting

$$\boldsymbol{u} = \boldsymbol{u}_0+\boldsymbol{u}_1,\quad p = p_0+p_1,\quad \rho = \rho_0+\rho_1 \tag{2.9}$$

and introduce the transformation

$$\boldsymbol{u}_1 = \rho_0^{-\frac{1}{2}}\boldsymbol{u},\quad (\rho_1,p_1) = \rho_0^{\frac{1}{2}}(\rho,p) \tag{2.10}$$

defining the field variables $\boldsymbol{u}$, ρ and p. From now onwards the symbols $\boldsymbol{u}$, ρ and p stand for field variables and should not be confused with the symbols used in the general basic equations, which will not appear again below. If we let $\boldsymbol{u} = (u, v, w)$ and assume that all field variables have the dependence

$$X(x,y,z,t) = \mathrm{Re}\{X(y)\exp\mathrm{i}(\omega t - kx - mz)\}, \tag{2.11}$$

we can write the perturbation equations in component form. Thus

$$\mathrm{i}\hat{\omega}u - (f - U')\,v = \mathrm{i}kp, \tag{2.12}$$

$$\mathrm{i}\hat{\omega}v + fu + fU\rho = \partial p/\partial y + \tfrac{1}{2}ap, \tag{2.13}$$

$$\mathrm{i}\hat{\omega}w = \mathrm{i}m_0 p - \rho g, \tag{2.14}$$

$$\mathrm{i}\hat{\omega}\rho = \mathrm{i}ku - (\partial/\partial y - \tfrac{1}{2}a)\,v + \mathrm{i}m_0 w, \tag{2.15}$$

$$\mathrm{i}\hat{\omega}p = (fU - ac^2)\,v + (g - c^2/H)\,w + \mathrm{i}\hat{\omega}\rho c^2. \tag{2.16}$$

The intrinsic (Doppler-shifted) frequency, $\hat{\omega}$, and the modified 'wavenumber' in the vertical direction, m_0, are defined by

$$\hat{\omega} = \omega - kU, \quad m_0 = m - \mathrm{i}/2H. \tag{2.17}$$

After lengthy, but straightforward, manipulations all variables can be eliminated in favour of p. Thus

$$p'' + \left\{\frac{2\mathrm{i}mfUN^2}{g(-\hat{\omega}^2 + N^2)} + \frac{Q}{(N^2-\hat{\omega}^2)}\left(\frac{N^2-\hat{\omega}^2}{Q}\right)'\right\}p' + \left\{\frac{Q}{\hat{\omega}(N^2-\hat{\omega}^2)}\left(\frac{E}{Q}\right)' - \frac{C}{N^2-\hat{\omega}^2}\right\}p = 0, \tag{2.18}$$

where

$$\left.\begin{aligned}
E &= kf(N^2-\hat{\omega}^2) + \hat{\omega}^3 fU\left(\frac{1}{2gH} - \frac{1}{c^2}\right) + \mathrm{i}\frac{m\hat{\omega}fUN^2}{g} = E_r + \mathrm{i}E_i,\\
Q &= (N^2-\hat{\omega}^2)\,(\hat{\omega}^2 - f^2 + fU') + \hat{\omega}^2 f^2U^2N^2/g^2 = -A(N^2-\hat{\omega}^2) + \hat{\omega}^2f^2U^2N^2/g^2,\\
C &= k^2(N^2-\hat{\omega}^2) + (k^2+m^2)\frac{f^2U^2N^2}{g^2} - \left(m^2 + \tfrac{1}{4}H^{-2} - \frac{\hat{\omega}^2}{c^2}\right)(\hat{\omega}^2 - f^2)\\
&\quad - \frac{\hat{\omega}^2f^2U^2}{4g^2H^2} - 2\hat{\omega}kUf\left(\frac{1}{c^2} - \frac{1}{2gH}\right) + U'f\left[\hat{\omega}kU\left(\frac{1}{c^2} - \frac{1}{2gH}\right)\right.\\
&\quad \left. - \left(m^2 + \frac{1}{4H^2} - \frac{\hat{\omega}^2}{c^2}\right)\right] - \mathrm{i}\frac{mkUfN^2U'}{g\hat{\omega}} = C_r + \mathrm{i}C_i.
\end{aligned}\right\} \tag{2.19}$$

The remaining field variables are related to p and its first derivative by

$$\left.\begin{aligned}
v &= \mathrm{i}Q^{-1}[Ep + \hat{\omega}(N^2-\hat{\omega}^2)\,p'],\\
\rho &= \frac{1}{Q}\left\{\left[\left(\frac{\hat{\omega}^2}{c^2} - \frac{\mathrm{i}m_0N^2}{g}\right)A - \hat{\omega}\left(f\frac{U}{c^2} - a\right)(\tfrac{1}{2}\hat{\omega}a - kf)\right]p + \hat{\omega}\left(f\frac{U}{c^2} - a\right)p'\right\},\\
u &= \frac{1}{Q}\left\{(f-U')fU\left[\hat{\omega}^2\left(\frac{1}{2gH} - \frac{1}{c^2}\right) + \frac{\mathrm{i}mN^2}{g}\right] + k\left(N^2 - \hat{\omega}^2 + f^2U^2\frac{N^2}{g^2}\right)\right\}p\\
&\quad + Q^{-1}(f-U')\,(N^2-\hat{\omega}^2)\,p',\\
w &= \frac{1}{Q}\left\{-\mathrm{i}g\left[\hat{\omega}A\left(\frac{1}{2gH} - \frac{1}{c^2}\right) + \frac{kf^2UN^2}{g^2}\right] + m\hat{\omega}\left(A + \frac{f^2U^2N^2}{g^2}\right)\right\}p - \mathrm{i}\frac{\hat{\omega}fUN^2}{Qg}p'.
\end{aligned}\right\} \tag{2.20}$$

In (2.12)–(2.20) a prime denotes differentiation with respect to the argument and N is the Brunt–Väisällä (buoyancy) frequency defined by

$$N^2 = g\left(\frac{1}{H} - \frac{g}{c^2}\right). \tag{2.21}$$

For the purpose of the discussions in the subsequent sections we require to make two transformations on p. First we take

$$p = h\phi, \quad h = [Q/(\hat{\omega}^2 - N^2)]^{\frac{1}{2}} \tag{2.22}$$

and use (2.18) to obtain the following equation for ϕ

$$(\hat{\omega}^2 - N^2)\,\phi'' - \mathrm{i}(2mfUN^2/g)\,\phi' + (b + \mathrm{i}d)\,\phi = 0, \tag{2.23}$$

in which

$$b = C_\mathrm{r} + \frac{(E_\mathrm{r}\,Q' - E'_\mathrm{r}\,Q)}{\hat{\omega}Q} + \frac{k^2N^2U'^2}{(\hat{\omega}^2 - N^2)} - kU'\hat{\omega}\frac{Q'}{Q} + kU''\hat{\omega} + Q''\frac{(\hat{\omega}^2 - N^2)}{2Q} - \tfrac{3}{4}Q'^2\frac{(\hat{\omega}^2 - N^2)}{Q^2},$$

$$d = -\frac{mN^2}{g}\left[\beta U + fU' + \frac{2kfUU'}{(\hat{\omega}^2 - N^2)}\right]. \tag{2.24}$$

The second transformation removes the coefficient of ϕ'. Thus letting

$$\phi = \chi\psi, \quad \chi'/\chi = \mathrm{i}mfUN^2/g(\hat{\omega}^2 - N^2), \tag{2.25}$$

and substituting into (2.23) we obtain

$$\psi'' + \mathscr{C}\psi = 0, \tag{2.26}$$

where

$$\mathscr{C} = \frac{b}{(\hat{\omega}^2 - N^2)} + \frac{m^2f^2U^2N^2}{g^2(\hat{\omega}^2 - N^2)^2}. \tag{2.27}$$

The equations (2.23) and (2.26) have been derived to ilustrate two important properties of the problem. We first note that $\mathscr{C}$ is real contrary to the findings in M.M. The discrepancy in the equation obtained in M.M. is that the term $fU\rho$ was neglected in the equation corresponding to (2.13) above. This resulted in the coefficient of ϕ', as in (2.23) above, being half its present value and thus $\mathscr{C}$ was found to be complex. Secondly, the dispersion relation of the system is obtained from (2.23) for real m and from (2.26) for imaginary m. The study in M.M. concentrated on the latter and we here deal with the former, although some new properties of the latter are also investigated. It transpires that the situation of real m gives rise to certain novel features like valve behaviour and over-reflexion by a finite shear. These phenomena will be discussed in §§ 3–6 below.

The equations (2.18), (2.23) and (2.26) governing linear wave motions in an isothermal atmosphere on a beta-plane in the presence of a latitudinally sheared zonal flow are all singular where

$$\hat{\omega} = 0, \quad \hat{\omega}^2 = N^2, \quad Q = 0, \tag{2.28}$$

which, in general, represent *seven* possible singular latitudes. However, in the Boussinesq approximation (i.e. in the limit $g \to \infty$, $H^{-1} \to 0$ with $H^{-1}g$ finite) the third equation of (2.28) reduces to $A = 0$ giving only *two* (instead of four) singularities because in this limit two of the singularities coincide with those given by the second of (2.28). Consequently the Boussinesq approximation filters out two singularities of the system. Another consequence of the Boussinesq approximation is that the coefficient of ϕ' in (2.23) would vanish. For a non-Boussinesq fluid however this term cannot be neglected in the neighbourhood of $\hat{\omega}^2 = N^2$ *no matter how small* $mN^2\phi'/g\phi''$ *may be.* The nature of the singularities (2.28) will be discussed in § 4 below.

3. The wave normal surfaces and ray trajectories

Before we proceed to study the full equations (2.23) and (2.26) it is constructive to carry out a W.K.B.J. treatment of these equations. This is valid provided that the wavelength λ of the waves in the northward direction is much smaller than the scale L of variations of U. In a formal W.K.B.J. (see, for example, Reid 1965) study the quantity $\lambda/L(=\epsilon)$ provides the small parameter but for our purposes here it suffices to set $U' = U'' = 0$ and assume solutions of the form

$$\phi(y) \propto \exp(-\mathrm{i}ly) \quad \text{or} \quad \psi(y) \propto \exp(-\mathrm{i}ny), \tag{3.1}$$

depending on whether (2.23) or (2.26) is to be solved, to obtain the dispersion relations

$$(\hat{\omega}^2 - N^2)\, l^2 + 2mfUN^2 l/g - b_1 = 0, \tag{3.2}$$

$$(\hat{\omega}^2 - N^2)\, n^2 - b_1 - m^2 f^2 U^2 N^2/[g^2(\hat{\omega}^2 - N^2)^2] = 0, \tag{3.3}$$

where

$$\begin{aligned} b_1 = {} & k^2\left(N^2 - \hat{\omega}^2 + \frac{f^2U^2N^2}{g^2}\right) + \left(m^2 + \frac{1}{4H^2} - \frac{\hat{\omega}^2}{c^2}\right)(f^2 - \hat{\omega}^2) + \frac{m^2 f^2 U^2 N^2}{g^2} \\ & - \frac{\hat{\omega}^2 f^2 U^2}{4g^2H^2} + \hat{\omega} k U f^2\left(\frac{N^2}{g^2} - \frac{1}{c^2}\right) - \frac{\beta(\hat{\omega}^2 - N^2)}{Q_1}\left\{\frac{k(\hat{\omega}^2 - N^2)(\hat{\omega}^2 + f^2)}{\hat{\omega}} - \frac{\hat{\omega}^2 f^2 U^2 N^2 k}{g^2}\right. \\ & \left. + \frac{1}{2}\left(\frac{N^2}{g^2} - \frac{1}{c^2}\right)\hat{\omega}^2 U(f^2 + \hat{\omega}^2) + \frac{\hat{\omega}^4 f^2 U^3 N^2}{2(\hat{\omega}^2 - N^2)}\left(\frac{N^2}{g^2} - \frac{1}{c^2}\right)\right\} \\ & - \frac{\beta^2(\hat{\omega}^2 - N^2)}{Q_1^2}\left[\hat{\omega}^2 - N^2 + \frac{\hat{\omega}^2 U^2 N^2}{g^2}\right]\left[(\hat{\omega}^2 - N^2)(2f^2 + \hat{\omega}^2) + \frac{2\hat{\omega}^2 f^2 U^2 N^2}{g^2}\right], \end{aligned} \tag{3.4}$$

in which

$$Q_1 = (\hat{\omega}^2 - N^2)(f^2 - \hat{\omega}^2) + \hat{\omega}^2 f^2 U^2 N^2/g^2. \tag{3.5}$$

The relations (3.2) and (3.3) are in general complicated and exact solutions require extensive numerical computations. It turns out however that for subacoustic wind profiles (i.e. when $U^2/c^2 \ll 1$) the problem can be studied analytically. Accordingly b_1 can be simplified to

$$\begin{aligned} b_1 = {} & k^2(N^2 - \hat{\omega}^2) + (m^2 + \tfrac{1}{4}H^{-2})(f^2 - \hat{\omega}^2) \\ & - \beta k(\hat{\omega}^2 - N^2)(f^2 + \hat{\omega}^2)/(Q_1\hat{\omega}) - \beta^2(\hat{\omega}^2 - N^2)^3(\hat{\omega}^2 + 2f^2)/Q_1^2. \end{aligned} \tag{3.6}$$

It should be remarked here that the last term in the expression for Q_1 (cf. (3.5)) is $O(U^2/c^2)$ but due to the possibility that $\hat{\omega}^2$ can take the values f^2, N^2 this term must be retained.

3.1. *The wave normal surfaces*

In the absence of a flow (i.e. $U = 0$) the relations (3.2) and (3.3) both reduce to

$$k^2 + l^2 + \frac{\beta k(f^2 + \omega^2)}{\omega(f^2 - \omega^2)} = \left(m^2 + \frac{1}{4H^2}\right)\left(\frac{f^2 - \omega^2}{\omega^2 - N^2}\right). \tag{3.7}$$

The propagation properties of this equation can better be seen through the alternative form

$$\left[k + \frac{\beta(f^2 + \omega^2)}{2\omega(f^2 - \omega^2)}\right]^2 + l^2 = \left(m^2 + \frac{1}{4H^2}\right)\left(\frac{f^2 - \omega^2}{\omega^2 - N^2}\right) - \frac{\beta^2(3\omega^4 + 6f^2\omega^2 - f^4)}{4\omega^2(f^2 - \omega^2)^2}, \tag{3.8}$$

which immediately shows that propagation is possible in the horizontal plane (i.e. real k and l) if

$$r^2 = \left(m^2 + \frac{1}{4H^2}\right)\left(\frac{f^2 - \omega^2}{\omega^2 - N^2}\right) - \frac{\beta^2(3\omega^4 + 6f^2\omega^2 - f^4)}{4\omega^2(f^2 - \omega^2)^2} > 0, \tag{3.9}$$

while propagation in vertical planes is always possible, although it may be restricted to small regions in certain circumstances, if $\omega^2 < f^2$ (if we assume that $f^2 < N^2$, as is the case in the Earth's atmosphere). When (3.9) is satisfied the cross-section of the wave normal surfaces in the (k, l) plane is a circle of radius r and centre at

$$(k_c, 0); \quad k_c = -\beta(f^2+\omega^2)/2\omega(f^2-\omega^2). \tag{3.10}$$

In M.M. (3.8) was studied when $\omega^2 \ll f^2 \ll N^2$ in which case eastward phase propagation is possible only if m is imaginary since (3.8) and (3.10) then take the form

$$\left.\begin{aligned}\left(k+\frac{\beta}{2\omega}\right)^2+l^2 &= \frac{\beta^2}{4\omega^2}-\frac{f^2}{N^2}\left(m^2+\frac{1}{4H^2}\right),\\ (k_c, 0) &\equiv (-\beta/2\omega, 0).\end{aligned}\right\} \tag{3.11}$$

Here we intend to study the propagation of waves for which $\omega^2 < N^2$ always assuming that $f^2 < N^2$. This range of values of ω^2 spans a wide spectrum of wave motions ranging from inertial waves (slightly modified by gravity) with periods of the order of a day to gravity waves (slightly modified by rotation) with periods of a few minutes.

For $f^2 < \omega^2 < N^2$ the centre of the circle (3.8) as given by (3.10) lies on the *positive* k-axis and propagation is possible if (3.9) is obeyed. Moreover westward phase propagation of these waves is possible if r is greater than the distance between the centre of the circle and the origin, i.e. if

$$s^2 = \left(m^2+\frac{1}{4H^2}\right)\left(\frac{\omega^2-f^2}{N^2-\omega^2}\right)-\frac{\beta^2(2f^2+\omega^2)}{(f^2-\omega^2)^2} > 0, \tag{3.12}$$

and then the circle will meet the l-axis at $l = \pm l_0$, where

$$l_0 = s, \tag{3.13}$$

the intersection of the circle and the k-axis occurring at $k = k_\pm$, where

$$k_\pm = -\frac{\beta(f^2+\omega^2)}{2\omega(f^2-\omega^2)} \pm r. \tag{3.14}$$

The group velocity in the (k, l) plane is defined by

$$\boldsymbol{u}_G = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}\right)$$

(cf. Lighthill 1965) and can be obtained directly from (3.8). Thus

$$\frac{\partial\omega}{\partial k} = -\left[k+\frac{\beta(f^2+\omega^2)}{2\omega(f^2-\omega^2)}\right]\Big/K(\omega), \quad \frac{\partial\omega}{\partial l} = -\frac{l}{K(\omega)}, \tag{3.15}$$

where
$$K(\omega) = \frac{\beta k(\omega^4+4f^2\omega^2-f^4)}{2\omega^2(f^2-\omega^2)^2}+\frac{\omega(f^2-N^2)(m^2+\frac{1}{4}H^{-2})}{(\omega^2-N^2)^2}+\frac{\omega\beta^2(5f^2+\omega^2)}{(f^2-\omega^2)^3}. \tag{3.16}$$

For $f^2 < \omega^2 < N^2$ the last two terms in the expression (3.16) are both negative while the first takes its maximum value at $k = k_\pm$. However when we substitute k_+ for k in (3.16) and use (3.14) we find that $K(\omega)$ is always negative, provided (3.9) is satisfied. Thus for $f^2 < \omega^2 < N^2$, the group velocity in the l-direction is always positive (negative) for positive (negative) l. (For illustration see figures 3 and 4.) This may be contrasted with the case $\omega^2 < f^2$, where the group velocity has the opposite sense.

In the presence of a flow U the cross-sections of the wave normal surfaces in (k, l) direction are more complicated. The most significant change is the appearance of asymptotes at $\hat{\omega} = 0, \pm N$. (The apparent asymptote at $Q_1 = 0$ is non-existent since no propagation is possible in the immediate neighbourhood of $Q_1 = 0$.) Realizing that if $\omega^2 < f^2$ it is still possible for certain values of k and U to have $\hat{\omega}^2 < f^2$ we shall study the full range $\hat{\omega}^2 \leqslant O(N^2)$. In this way it is found that the restrictions

(i) $\hat{\omega}^2 \ll f^2$,
(ii) $\hat{\omega}^2 \ll N^2$,
(iii) Boussinesq approximation,

made in M.M. from the outset filtered out certain characteristic features exhibited by the propagation of Rossby-gravity waves on a beta-plane in the presence of latitudinally sheared zonal flows (see figures 1–9). Prominent among these is the valve behaviour at the critical latitudes $\hat{\omega}^2 = N^2$ when m is real.

Consider (3.2) and (3.4) in the vicinity of $\hat{\omega}^2 = N^2$. They can be solved for l to give

$$\left.\begin{aligned} l_f &\approx \left(m^2 + \frac{1}{4H^2}\right)\frac{(f^2 - N^2)\,g}{2mfUN^2}, \\ l_\infty &\approx -\frac{2mfUN^2}{g(\hat{\omega}^2 - N^2)}, \end{aligned}\right\} \tag{3.17}$$

where l_f refers to the finite value of l at $\hat{\omega}^2 = N^2$ and l_∞ refers to the value that increases indefinitely as $\hat{\omega}^2$ approaches N^2 (see figures 1–4). The group velocity in the l-direction (for $\hat{\omega}^2 \approx N^2$) is given by

$$\frac{\partial \omega}{\partial l} = v_G = \frac{-(\hat{\omega}^2 - N^2)\,l - mfUN^2/g}{\hat{\omega}(k^2 + l^2 + m^2 + \frac{1}{4}H^{-2})}. \tag{3.18}$$

When $l \to \infty$, we have, after using (3.17),

$$v_G \approx mfUN^2/(g\hat{\omega}l_\infty^2), \tag{3.19}$$

an expression that tends to zero as $\hat{\omega}^2$ approaches N^2. The ray (3.19) then approaches the critical latitude, y_c, in a way governed by the equation

$$\partial y/\partial t = -a(y - y_c)^2, \tag{3.20}$$

where $a\ (\neq 0)$ is a prescribed constant (see, for example, McKenzie 1973). Thus

$$y - y_c = (b + at)^{-1} \tag{3.21}$$

and therefore the ray, whose position is y, cannot reach the critical latitude y_c in a finite time. Thus this ray is absorbed at the critical latitute.

On the other hand,

$$v_{Gf} \approx \frac{-mfUN^2/g}{\hat{\omega}(l_f^2 + k^2 + m^2 + \frac{1}{4}H^{-2})} \tag{3.22}$$

at $\hat{\omega}^2 = N^2$ and it is finite there. This means that the ray (3.22) propagates across the critical latitude $\hat{\omega}^2 = N^2$ unhindered. Thus a wave approaching any one of the critical latitudes $\hat{\omega}^2 = N^2$ is transmitted or captured according to whether

$$gmfU\hat{\omega}v_G \lessgtr 0. \tag{3.23}$$

Such valve behaviour was first detected by Acheson (1972) and McKenzie (1973) for hydromagnetic waves and by Acheson (1973) for non-hydromagnetic waves. It has since been found that such behaviour is suffered by a variety of critical levels (see, for example, Grimshaw 1975, Eltayeb & Kandaswamy 1979).

Another characteristic feature concerns the 'coupling' between planetary and gravity waves. Here a planetary wave in a stationary medium (or in a medium with a weak zonal flow) approaching a shear layer may emerge as a gravity wave on the far side of the shear provided the shear is strong enough. This fact will be shown in § 6 below to effect over-reflexion.

The above results (and also for frequent use in the discussions in §§ 4–7 below) can conveniently be illustrated geometrically. We shall now summarize a few elementary results to help us follow the evolution of the wave normal surfaces with the flow U.

If $s^2 > 0$ (see (3.12) above), the wave normal surfaces will meet the l-axis in $l_\pm$, where

$$l_\pm = \frac{mfUN^2}{g(N^2-\hat{\omega}^2)} \pm s. \tag{3.24}$$

When $l = 0$, the surfaces meet the k-axis at points satisfying

$$b_1 = 0, \tag{3.25}$$

where b_1 is given by (3.6). If $|U|$ is small then (3.25) yields the roots

$$k_1 = \frac{\omega}{U}+\frac{\beta}{\omega}, \quad k_{2,3} = \frac{\omega \mp N}{U} + (m^2+\tfrac{1}{4}H^{-2})\frac{(N^2 \mp f^2)\,U}{2N(\omega \mp N)^2}, \tag{3.26}$$

in addition to the roots $k_\pm$, as given by (3.14), but they are now slightly modified by the presence of the flow (provided $r^2 > 0$). If $r^2 < 0$ then only the roots (3.26) are present. For large values of $|U|$, (3.25) has only three roots given by

$$k_1 = \frac{\omega}{U}+\frac{\omega f^2 N^2 \beta}{U^2[f^2(m^2+\frac{1}{4}H^{-2})+2\beta^2N^2]},$$

$$k_2 = \frac{\omega-\alpha}{U}, \quad k_3 = \frac{\omega+\alpha}{U}, \quad |U| \to \infty, \tag{3.27}$$

in which α^2 is the only positive root of the cubic

$$(m^2+\tfrac{1}{4}H^{-2})(f^2-\hat{\omega}^2) - \beta^2(\hat{\omega}^2-N^2)(2f^2+\hat{\omega}^2) = 0, \tag{3.28}$$

which satisfies the inequality

$$f^2 < \alpha^2 < N^2. \tag{3.29}$$

Also, asymptotes exist at $k = k_\infty, k_{\infty\pm}$ where

$$k_\infty = \frac{\omega}{U}, \quad k_{\infty\pm} = \frac{\omega \pm N}{U}. \tag{3.30}$$

The evolution of the cross sections of the wave normal surfaces in horizontal planes with variations in the basic flow is summarized in figures 1–4.

The dispersion relation (3.2) also governs the propagation of Rossby-gravity waves in a *vertically* sheared zonal flow if the vertical wavelength is much smaller than the vertical variations of the basic flow. It is then possible to investigate the cross-sections of the wave normal surfaces in the (m, k) plane for fixed values of l. Examination of the dispersion relation (3.7) and (3.8) shows that propagation is possible (i.e. real m) only if either ω^2 is less than $f^2 (< N^2)$ in which case k must be negative or if $f^2 < \omega^2 < N^2$ and then k can be either positive or negative. The wave normal curves in the (k, m) plane are ellipses if $\omega^2 < f^2$ and hyperbolae if $f^2 < \omega^2 < N^2$.

In the presence of a flow the wave normal curves in (k, m) planes possess three asymptotes (i.e. critical heights) at

$$\hat{\omega} = 0, \quad \pm f. \tag{3.31}$$

By the use of other elementary properties, as in the case of the wave normal curves in (l, k) planes, these curves are illustrated in figures 5 and 6.

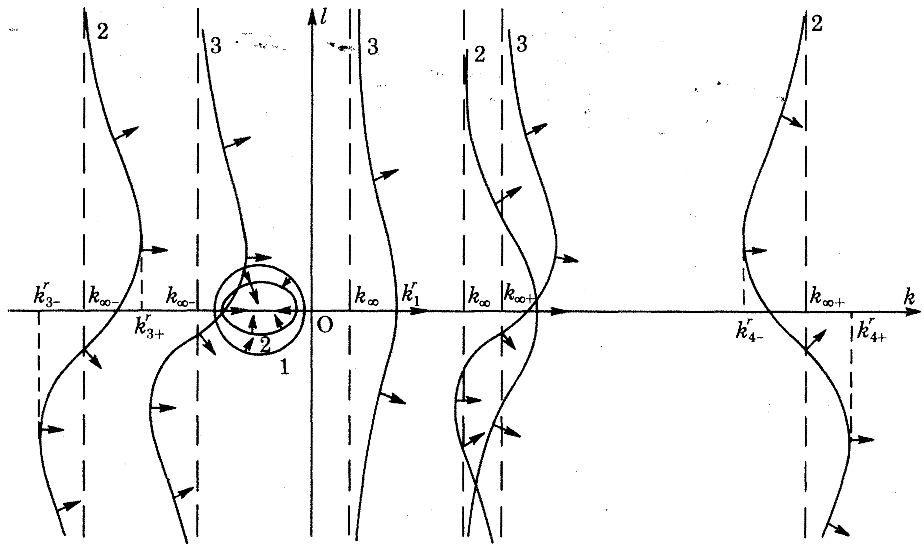

FIGURE 1. The cross-section of the wave normal surfaces in the (k, l) plane for increasing values (counting up) of a westerly flow $(U > 0)$ when $\omega^2 < f^2 < N^2$ and m is real. The circle 1 is present in the absence of the flow if (3.9) is satisfied. As U increases it shrinks until it disappears at a value $\bar{U}$ of U. If (3.9) is not satisfied the circle 1 is absent and so is the closed loop 2, but the other curves remain. For notation see § 3.

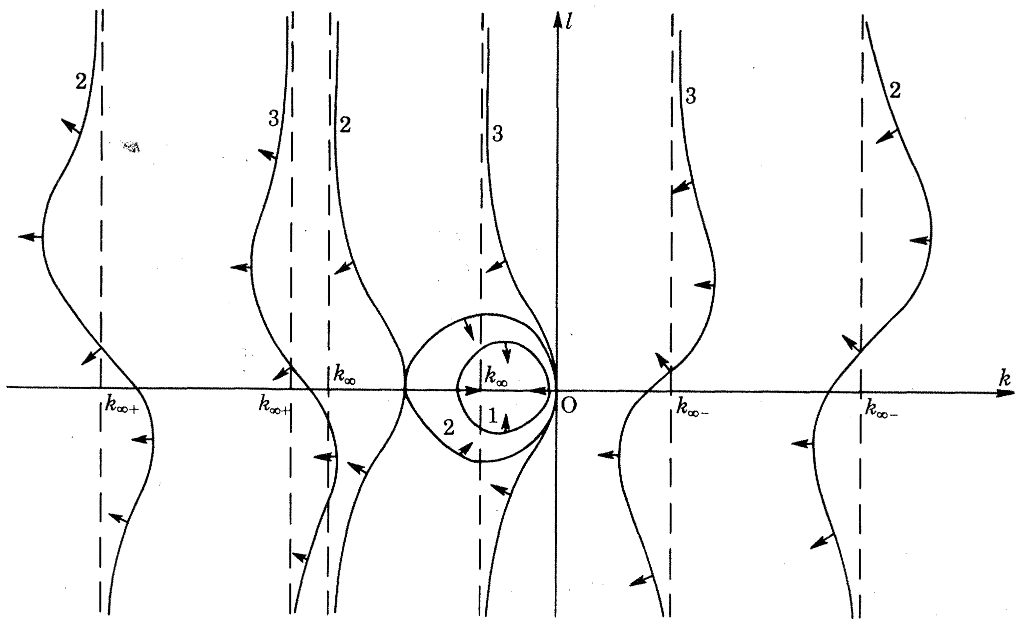

FIGURE 2. The wave normal curves in the (k, l) plane for increasing values (counting up) of an easterly flow when $\omega^2 < f^2 < N^2$ and m is real. The closed loops 1 and 2 are present only if (3.9) is obeyed.

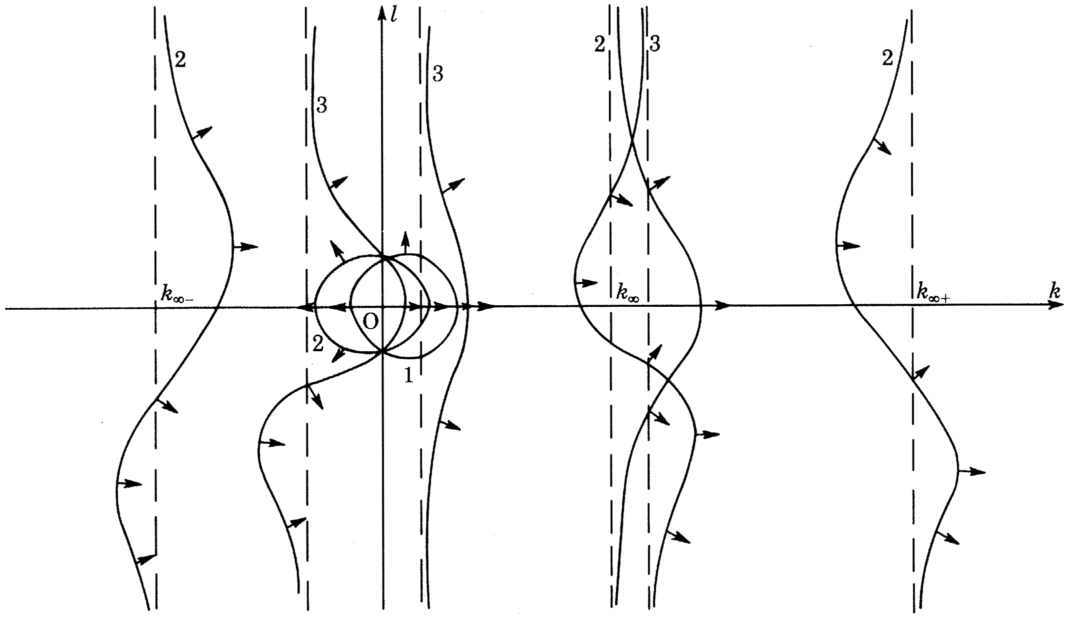

FIGURE 3. The evolution of the wave normal curves in the (k, l) plane with increasing values of a westerly wind (counting up) for $f^2 < \omega^2 < N^2$ and real m. When (3.9) is satisfied the circle 1 is pushed to the left until it coalesces with the $k_{\infty-}$-branch of the gravity wave as in stage 3. If (3.9) is violated then the closed loops are missing and the $k_{\infty-}$-branch lies wholly to the left of the l-axis. The points at which the curves meet the l-axis, when they exist, will vary with U if U is large enough to compare with c.

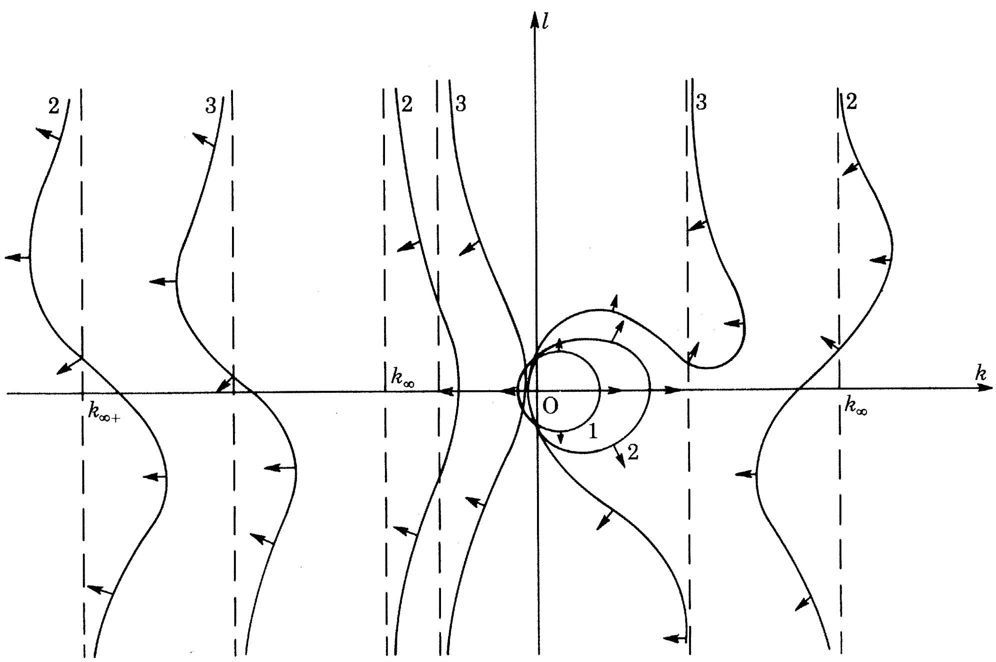

FIGURE 4. The same as figure 3 except that the flow is easterly. The figure is drawn for $s^2 > 0$ (cf. equation (3.12)). If $s^2 < 0$ but $r^2 > 0$ (cf. equation (3.9)) the closed loops will lie on the right of the l-axis and will disappear completely if $r^2 \leqslant 0$.

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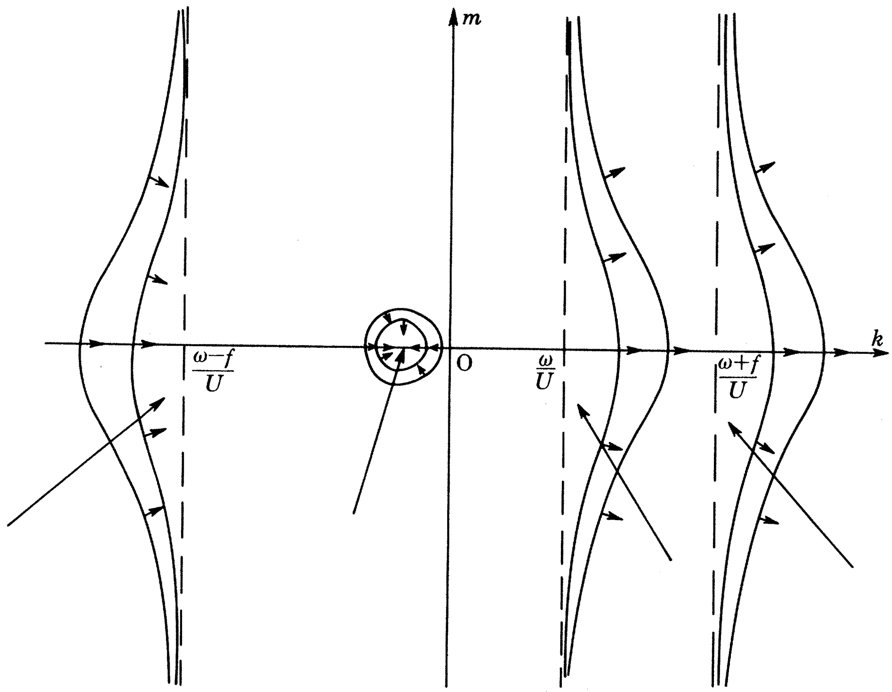


FIGURE 5. The cross sections of the wave normal surfaces in the (k, m) plane for various values of l when $\omega^2 < f^2 < N^2$. The long arrows give the direction of increase of $|l|$ and the short arrows give the direction of group velocity. The closed loop disappears for large $|l|$.

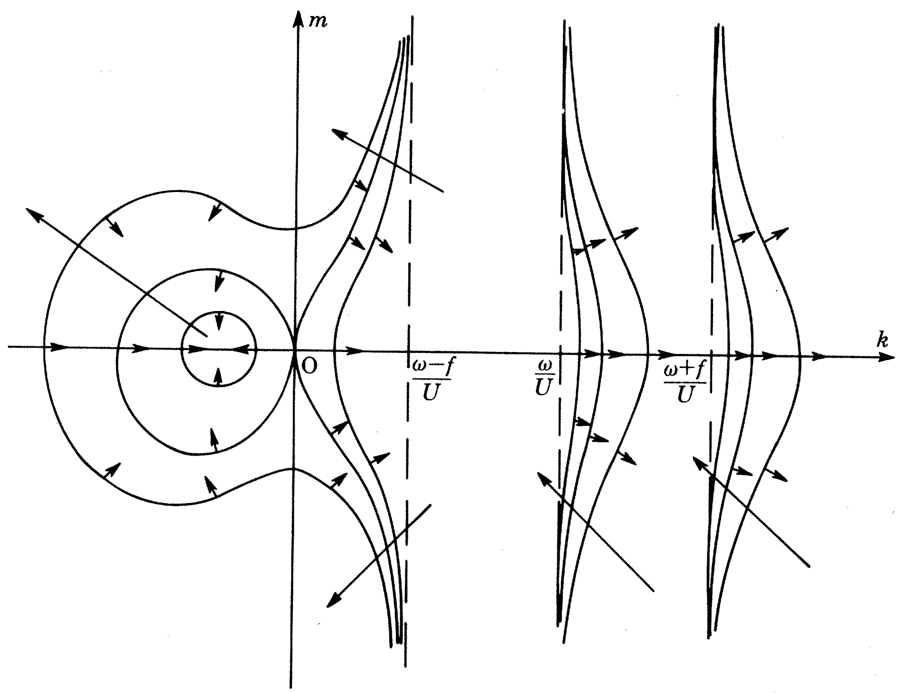


FIGURE 6. The same as in figure 5 except that here $f^2 < \omega^2 < N^2$.

3.2. *Ray trajectories*

The dispersion relation (3.2) can be written in the functional form

$$\omega(k, l, m;\ N, f, \beta, U, g) = 0. \tag{3.32}$$

For fixed values of the parameters, N, f, β, U, g this relation represents a surface S in the (k, l, m) space. The gradient to S is the group velocity

$$\boldsymbol{u}_{\mathrm{G}} = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}\right) = (u_{\mathrm{G}}, v_{\mathrm{G}}, w_{\mathrm{G}}), \tag{3.33}$$

which represents the direction of the ray at that locality. Moreover, the zonal wavenumber k and frequency ω are conserved along a ray path in the (x, y) plane (Longuet-Higgins 1964*b*; Lighthill 1967*b*). By locating the direction of the group velocity, which is normal to S by (3.33), at every point it is then possible to construct the various types of ray trajectory that can occur in different profiles of zonal flow.

Now the group velocity $(u_{\mathrm{G}}, v_{\mathrm{G}})$ in the (x, y) plane is given by

$$\left.\begin{aligned} u_{\mathrm{G}} &= U + (N^2-\hat\omega^2)\,[k+\beta(\hat\omega^2-N^2)\,(f^2+\hat\omega^2)/(\hat\omega Q_1)]/K(\hat\omega),\\ v_{\mathrm{G}} &= -[l(\hat\omega^2-N^2)+mfUN^2/g]/K(\hat\omega)\end{aligned}\right\} \tag{3.34}$$

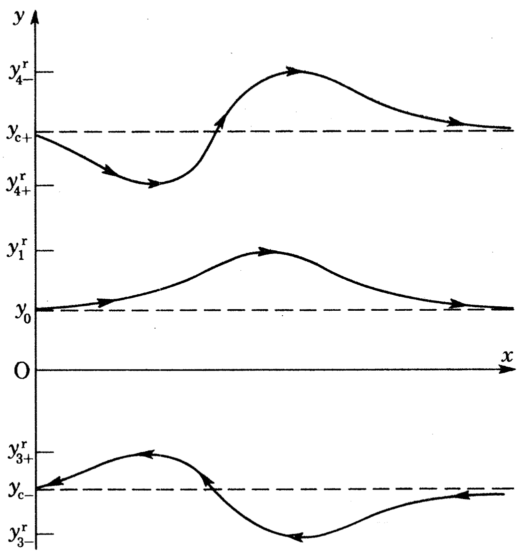
in which

$$\begin{aligned} K(\hat\omega) = {} & \hat\omega(k^2+l^2+m^2+\tfrac{1}{4}H^{-2}) + \beta k(N^2-\hat\omega^2)\,(2f^2+3\hat\omega^2-N^2)/Q_1^2\\ & +[\beta k(N^2-\hat\omega^2)\,(f^2+\hat\omega^2)\,(N^2+f^2-2\hat\omega^2+f^2U^2N^2/g^2)\\ & -\beta^2(N^2-\hat\omega^2)\,(6f^2+4\hat\omega^2-N^2)\,\hat\omega]/Q_1^2\\ & -2\hat\omega\beta^2(2f^2+\hat\omega^2)\,(N^2+f^2-2\hat\omega^2+f^2U^2N^2/g^2)/Q_1^3\end{aligned}$$

when m is real. A ray propagating horizontally will be reflected at points where $v_{\mathrm{G}} = 0$, together with the conditions that l is finite and governed by the dispersion relation there. For each value of $U\,(\neq 0)$ five reflexion points are found if no propagation is possible in the absence of a flow and seven otherwise (see figures 1–4). Let us adopt a notation for these points. We denote the reflexion points in the absence of a flow (if they exist) by $k^{\mathrm{r}}_{\pm}$, those near $k_{\infty-}$ by $k^{\mathrm{r}}_{3\mp}$ (with $k^{\mathrm{r}}_{3-} < k^{\mathrm{r}}_{3+}$), those near $k_{\infty+}$ by $k_{4\mp}$ and that one near k_{∞} by k^{r}_{1} (see figure 1). For a particular wave (i.e. given k, m, ω, N, f, β, g) these reflexion points can be encountered by the wave only if U attains a certain value. We shall give such U the same label as that of k_{r}, i.e. if a wave reaches a latitude corresponding to k^{r}_{1} we shall denote the value of U there by U^{r}_{1} and the value of y there by y^{r}_{1}.

In figures 7–9 we sketch some of the various types of ray trajectory that can arise in an anti-symmetric profile of U. Here the planetary wave trajectories lie in the vicinity of $U = 0$ while the gravity wave rays (with their valve behaviour) occur far away from $y = 0$ where the flow is strong.

In symmetric jet-like streams the inclusion of gravity waves to the treatment in M.M. introduces novel features. In particular for planetary waves (i.e. $\omega^2 < f^2$ and $k_- < k < k_+$) the rays in the vicinity of the centre of the jet are strongly influenced by the value of the speed at the jet-centre, $U_{\max}$. For small values of $U_{\max}$ (assuming the jet is westerly) planetary waves can penetrate the centre of the jet, as has been shown in M.M. (cf. figures 10, 11 and 12). If however $U_{\max}$ is increased planetary waves are expelled from the jet centre (essentially because $\hat\omega^2$ now exceeds f) and gravity wave rays appear there. Indeed for certain values of $U_{\max}$ two critical latitudes, of the gravity wave type, appear, one on either side of the jet centre. Provided $U_{\max}$



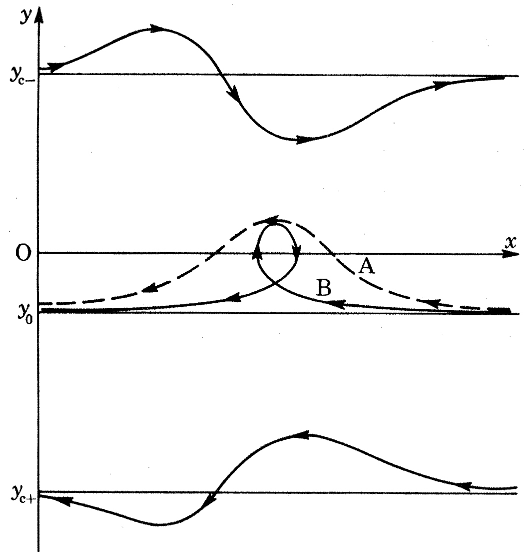


FIGURE 7. The ray system in a westerly wind increasing steadily from $-\infty$ at $y = -\infty$ to $+\infty$ at $y = +\infty$, being zero at $y = 0$, for $k > 0$ in the case $\omega^2 < f^2 < N^2$ and m $(\neq 0)$ is real. If m is imaginary only the branch near y_0 remains and if $m = 0$ the valve behaviour at $y_{c\pm}$ disappears and the rays there become similar to the one near y_0 with both rays lying on the sides nearest to y_0. Note that here and in the following figures y_0 refers to the R.w.c.l. and $y_{c\pm}$ refer to the g.w.c.ls at $k_{\infty\pm}$.

FIGURE 8. The same as figure 7 but for $k < 0$. Near y_0 the looped ray B corresponds to $k_c < k < k_+$ and the discontinuous ray A corresponds to $k < k_c$, when $k_\pm$ are real. If $k_\pm$ do not exist then ray A is present. The figure is drawn for m $(\neq 0)$ real. For $m = 0$ the same modifications as in figure 7 apply.

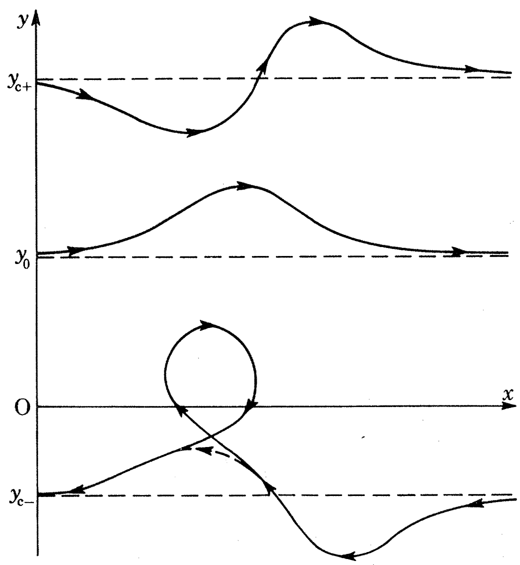


FIGURE 9. The ray system for the flow specified in figure 7 when $f^2 < \omega^2 < N^2$ for real non-zero m and positive k. The loop is relevant if $0 < k < k_0$. The reflexion points have the same meaning as in figure 7.

is not too great (or more precisely if $(\omega-N)/k < U_{max} < U^{r}_{3-}$) a wave propagating towards a critical latitude may be transmitted to the other side, which is nearest to the jet-centre, propagate across the jet centre and advance towards the other critical latitude across which it is transmitted to emerge in the far wing of the jet. In the same circumstances a wave is trapped between the two critical latitudes around the centre of the jet (see figures 10 and 11). Similar behaviour occurs in an easterly jet (see figures 12 and 13). Moreover, in an easterly jet, and provided $f^2 < \omega^2 < N^2$ and $0 < k < k_+$, for certain values of U_{max} a wave can propagate from the centre of the jet towards a gravity wave critical latitude which it crosses to propagate far into the wing of the jet (figure 13*d*). Furthermore, situations may arise whereby a wave propagating from one wing of the jet may pass through the jet centre and right across a critical latitude to emerge on the far wing of the jet. Some of these situations are illustrated in figure 14.

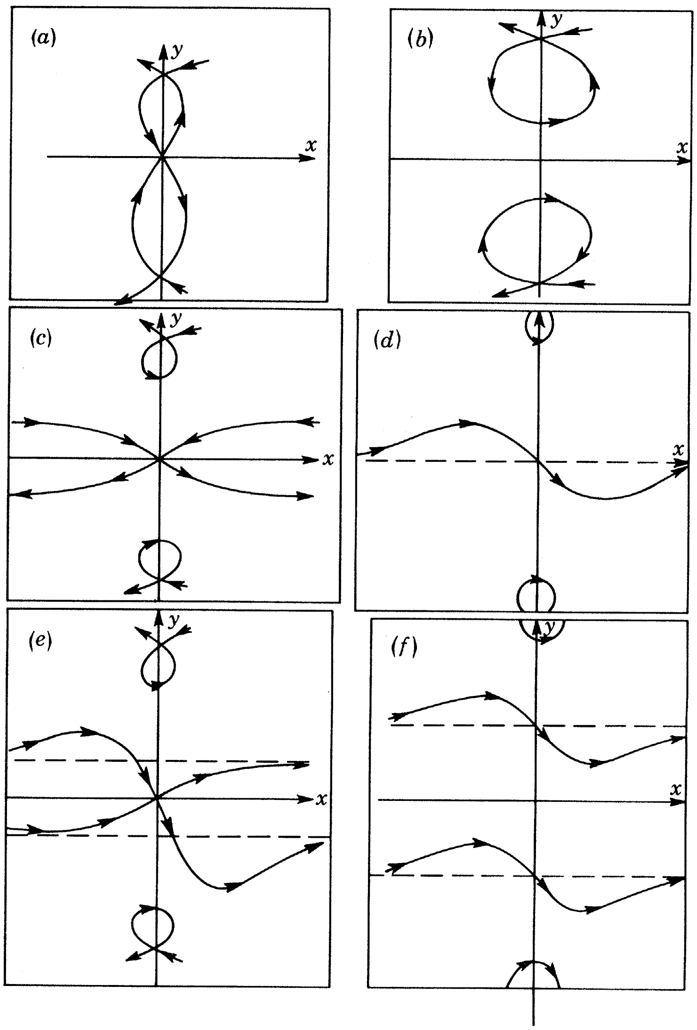


FIGURE 10. The evolution of the system of rays in a westerly jet-like wind when $\omega^2 < f^2 < N^2$ for increasing values of the maximum jet speed for values in the range $k_- < k < k_c$. The origin of coordinates is taken at the centre of the jet. If U_d is the smallest value of U at which equation (3.2) has a repeated root for l and U_{max} is the speed at the jet centre then the figure is classified as follows: (*a*) $U_{max} < U_d$; (*b*) $U_d \leqslant U_{max} < U^{r}_{3+}$; (*c*) $U^{r}_{3+} < U_{max} < (\omega-N)/k$; (*d*) $U_{max} = (\omega-N)/k$; (*e*) $(\omega-N)/k < U_{max} < U^{r}_{3-}$; (*f*) $U_{max} > U^{r}_{3-}$. Note that (*a*) corresponds to figure 5(*d*) in M.M.

Before we conclude this section we briefly discuss the normalized dispersion relation (3.3). This relation represents the dispersion relation of the system when m is imaginary in which case the wave normal curves in the (l, k) plane are the same as those studied in M.M. (see figure 9 in M.M.), because no propagation can then take place near the singularity $\hat{\omega}^2 = N^2$. Moreover, the curves in the (n, k) plane associated with real m are required for the study on the reflexion by a finite shear layer carried out in § 6 below. For this purpose we have sketched these curves for $\omega^2 < f^2$ in figure 15 which also defines the notation required in § 6.

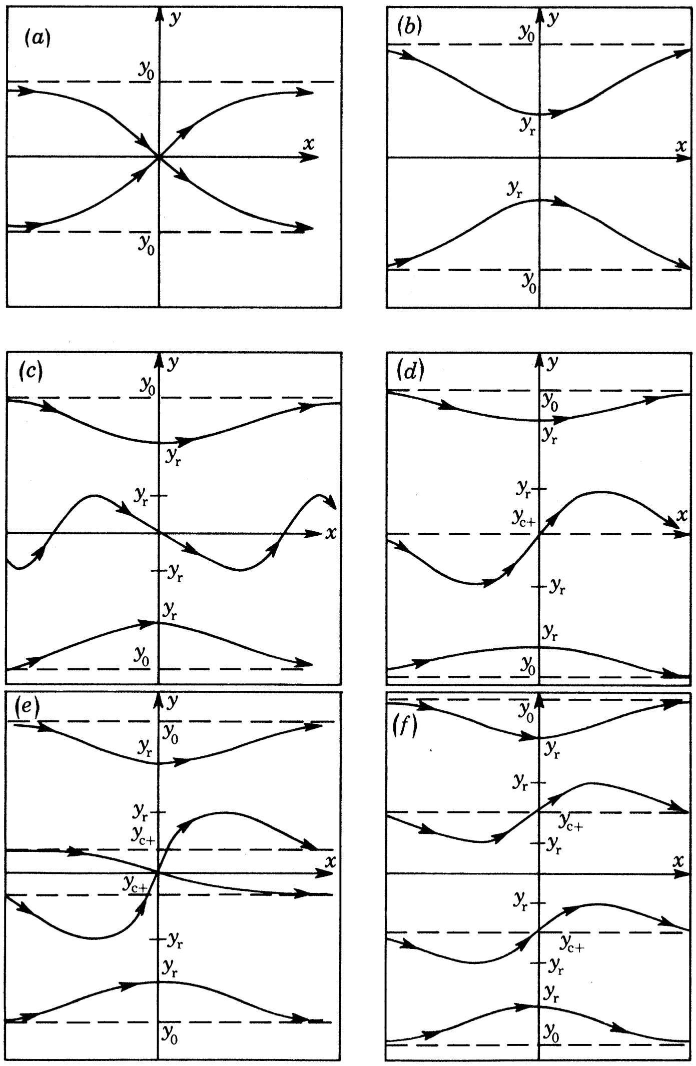


FIGURE 11. The rays that can arise in a westerly jet stream when $f^2 < \omega^2 < N^2$ and $k > k_+$ for increasing values of the speed at the centre of the jet. If $U_{max} < \omega/k$ no propagation takes place. (a) $\omega/k \leqslant U_{max} < U_1^r$; (b) $U_1^r < U_{max} \leqslant U_{4-}^r$; (c) $U_{4-}^r < U_{max} < (\omega+N)/k$; (d) $U_{max} = (\omega+N)/k$; (e) $(\omega+N)/k < U_{max} \leqslant U_{4+}^r$; (f) $U_{max} > U_{4+}^r$.

Another remark concerns the case $m = 0$, i.e. two-dimensional wave motions. In this case propagation of waves of the gravity type (i.e. $\hat{\omega}^2 \approx N^2$) can occur only on one side of a gravity wave critical latitude. It appears then that the situation here is similar to that of Rossby wave critical latitude but a detailed study employing a full wave treatment (§ 5 below) shows that even in this case the gravity wave critical latitude is different from a Rossby wave critical latitude.

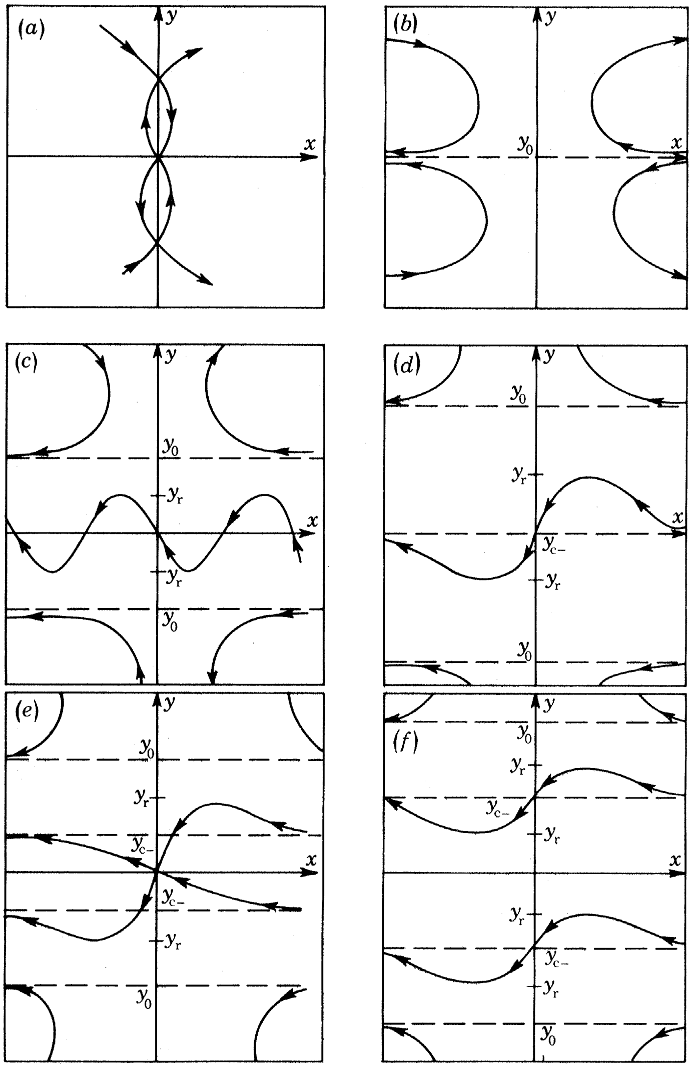


FIGURE 12. The rays arising in an easterly jet-like wind when $k_- < k < k_c$ in the range $\omega^2 < f^2 < N^2$ for different values of maximum jet speed $-U_{\min}(U < 0)$: (a) $U_{\min} > U_0$; (b) $U_{\min} = U_0$; (c) $U_0 > U_{\min} > (\omega + N)/k$; (d) $U_{\min} = (\omega + N)/k$; (e) $(\omega + N)/k > U_{\min} \geqslant U^r_{4-}$; (f) $U_{\min} < U^r_{4-}$.

4. Energetics of the system

The investigation of the wave normal surfaces for Rossby-gravity waves in the preceding section indicated the existence of critical latitudes. Before we proceed to examine the present system in the neighbourhoods of these critical latitudes we require knowledge of some of the basic physical quantities like momentum transfer, energy flux and energy density so that the physical nature of these latitudes can be assessed. It transpires that these quantities can best be dealt with in terms of the wave-invariant of the system.

Since $\mathscr{C}$ in equation (2.26) is real, it follows from Eltayeb (1977, § 2) that the quantity

$$\mathscr{A} = \operatorname{Im}(\psi^* \mathrm{d}\psi/\mathrm{d}y), \tag{4.1}$$

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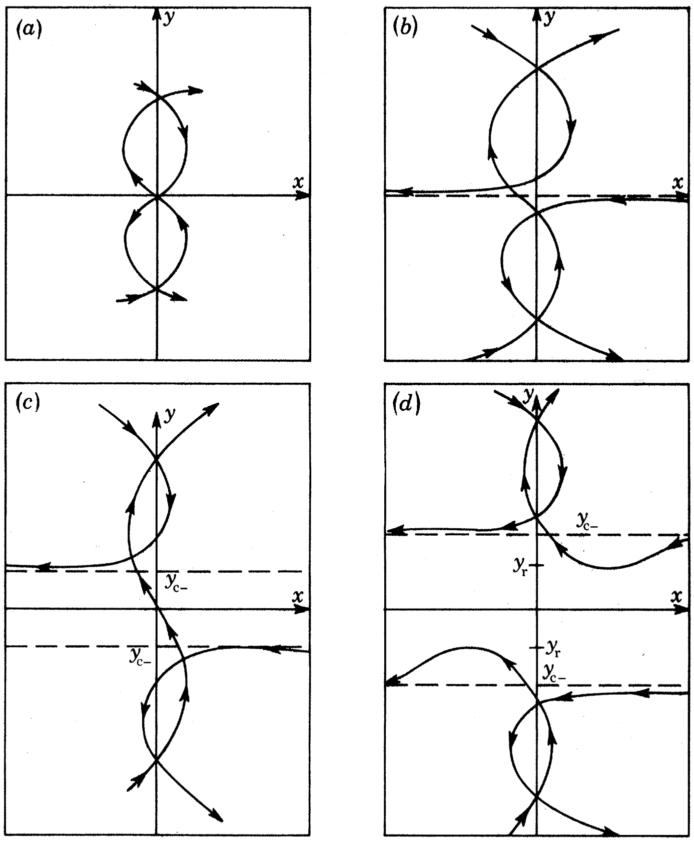

FIGURE 13. The evolution of the rays with the increase of speed at the centre of an easterly jet when $k_c < k < k_+$ and $f^2 < \omega^2 < N^2$: (a) $U_{min} > (\omega - N)/k$; (b) $U_{min} = (\omega - N)/k$; (c) $(\omega - N)/k > U_{min} \geqslant U^r_{3+}$; (d) $U_{min} < U^r_{3-}$. Note that on either side of the jet centre the ray propagating away from the jet centre manages to escape to the wing of the jet even if it started on the inside of a critical latitude.

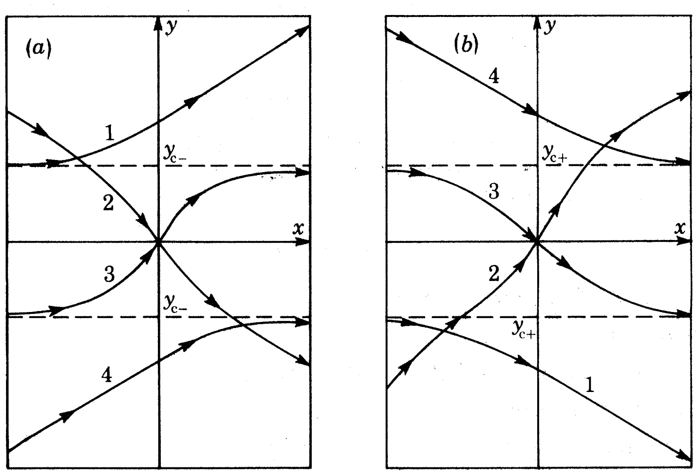

FIGURE 14. The ray systems in (a) a westerly jet stream corresponding to figure 10(e) when the flow at the wings of the jet exceeds $(\omega - N)/k$ and (b) an easterly jet stream corresponding to figure 12(e) when the flow at the wings of the jet has a magnitude greater than $|(\omega + N)/k|$. In both cases four types of rays are exhibited: (i) Rays 1 propagate from the vicinity of one of the critical latitudes towards the nearest wing of the jet. (ii) Rays 2 propagate from one wing of the jet towards the centre. They penetrate the first critical latitude they encounter and propagate right through the jet centre, across the second critical latitude on the other side of the jet centre and deep into the far wing of the jet. These rays can transfer energy and momentum from one wing of the jet to the other. (iii) Rays 3 propagate from the neighbourhood of a critical latitude across the centre of the jet to be absorbed at the other critical latitude. (iv) Rays 4 propagate from the wing of the jet towards a critical latitude where they suffer absorption. These rays transport energy and momentum from the wings to the centre of the jet.

where the asterisk denotes the complex conjugate, is independent of y. It will be shown presently that $\mathscr{A}$ is closely related both to the northward flux of wave energy and the northward transfer of zonal momentum. The northward transfer of zonal momentum is defined by

$$M = \rho_0 \overline{u_1 v_1}, \tag{4.2}$$

where the overbar denotes an average over a period. By using the field variables and the expression (2.20) we can write M in the form

$$M = (k/2\hat{\omega})(pv^* + p^*v) = k|\chi|^2\mathscr{A}, \tag{4.3}$$

after having used the transformations (2.22) and (2.25).

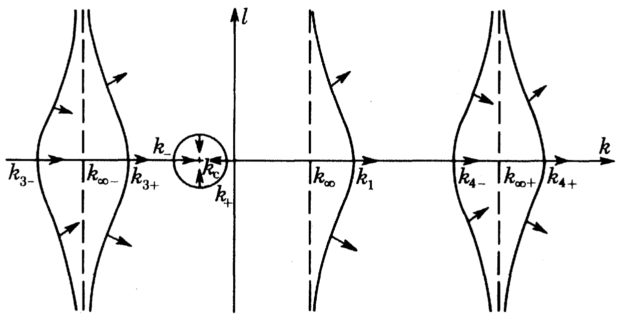

FIGURE 15. A sketch of the relation (3.3) in the (n, k) plane for a situation corresponding to stage 2 of figure 1. If $m = 0$ only the branches (at $k_{\infty\pm}$) nearest to the n-axis are present and those on the far sides disappear. For imaginary m all the branches near $k_{\infty\pm}$ disappear.

The northward eddy flux of energy (i.e. the northward energy flux as measured by an observer moving with the flow) is

$$F_e = \overline{p_1 v_1} = \overline{pv} = \tfrac{1}{2}(pv^* + p^*v) = \hat{\omega}|\chi|^2\mathscr{A}. \tag{4.4}$$

The total wave energy flux in the northward direction (i.e. the northward energy flux as measured by a *stationary* observer) is

$$F = \overline{p_1 v_1} + \overline{\rho_0 U u_1 v_1} = \overline{pv} + U\overline{uv}$$
$$= (\omega/\hat{\omega})(pv^* + p^*v) = \omega|\chi|^2\mathscr{A}. \tag{4.5}$$

The expressions (4.3) and (4.5) obey the classical relation

$$F = (\omega/k)\, M, \tag{4.6}$$

already obtained in M.M. However, the relation of $\mathscr{A}$ to M and F depend on the value of m. If m is real then $|\chi| = 1$ and hence the invariance of $\mathscr{A}$ represents the conservation of both the northward transfer of momentum and the total northward wave energy flux. If m is imaginary however, neither M nor F is conserved. Had the Boussinesq approximation been adopted, as in M.M., $|\chi|$ would not have appeared in (4.3) or (4.5) and M and F would have been conserved even if m was imaginary. Whether m is real or imaginary the relations (4.3) and (4.5) provide a means of studying the physical implications of the critical latitudes at $\hat{\omega} = 0, \pm N$. This will be carried out in § 5 below.

The total energy density as defined by

$$\mathscr{E} = \tfrac{1}{2}[uu^* + vv^* + ww^* + U(\rho u^* + \rho^* u)], \tag{4.7}$$

is found to be a complicated expression of ψ and ψ' and because of the appearance of a term proportional to $|\psi'|^2$ it was not possible to relate it directly to $\mathscr{A}$. It may be possible, however, to define a local group velocity (which would again be complicated) to relate $\mathscr{E}$ to F. We will not pursue this any further here.

5. Solutions near critical latitudes

The wave equation (2.26) is singular at

$$\hat{\omega} = 0, \tag{5.1}$$

$$\hat{\omega}^2 = N^2, \tag{5.2}$$

$$Q = 0. \tag{5.3}$$

The purpose of this section is to investigate the solutions near each of these seven singularities and examine the influence of each singularity on the northward energy flux and momentum transfer. We shall find it convenient to adopt the notation that $\hat{\omega} = 0$ denotes a Rossby wave critical latitude, $\hat{\omega}^2 = N^2$ refers to gravity wave critical latitudes and $Q_1 = 0$ denotes gravity-inertial wave singularities.

5.1. *Rossby wave critical latitudes*

Here the wave equation takes the form

$$\psi'' + \frac{\alpha}{(y-y_0)}\psi = 0, \quad \alpha = \frac{(\beta - U_0'')}{U_0'}\frac{(f-U_0')}{f}, \tag{5.4}$$

in the neighbourhood of the critical latitude y_0, where y_0 is defined by

$$U(y_0) = \omega/k. \tag{5.5}$$

The solution of (5.4) was obtained in M.M. Thus

$$\psi = \begin{cases} A[1-\alpha(y-y_0)\ln(y_0-y)] + B(y-y_0) & \text{for} \quad y < y_0 \\ A[1-\alpha(y-y_0)\{\ln(y-y_0) \mp \mathrm{i}\pi\}] + B(y-y_0) & \text{for} \quad y > y_0. \end{cases} \tag{5.6}$$

The relation

$$\mathscr{A}_\mathrm{n} = \mathscr{A}_\mathrm{s} \pm \pi\alpha|A|^2 \quad \text{for} \quad kU_0' \gtrless 0, \tag{5.7}$$

in which n denotes north of (i.e. $y > y_0$), and s stands for south of (i.e. $y < y_0$) the critical latitude y_0, was also obtained in M.M.

In slowly varying shear U_0' and U_0'' are small and consequently α is approximately equal to (β/U_0') so that propagation takes place on the $y > y_0$ side if U_0 is increasing and on the side $y < y_0$ if U_0 is decreasing (see figures 1–4). In a full wave treatment, however, α can attain positive values even if $U_0' < 0$. Indeed if $U_0' < 0$ and $U_0'' > \beta$ then α is positive (for positive f). Thus in a full wave treatment propagation will occur on one side or the other depending on U_0' and on the gradient of the potential vorticity (i.e. on $\beta - U_0''$) at the critical level. The influence of the potential vorticity gradient on the wave in the vicinity of the R.w.c.l. has already been discussed by Geisler & Dickinson (1974). The foregoing discussion indicates that the W.K.B.J. solutions near the R.w.c.l. cannot be expected to provide a reasonable (even) qualitative picture of the real behaviour of the waves there.

However, the relation (5.7) provides a reasonable description of the state of affairs near the R.w.c.l. Since $|\chi|$ is a constant to leading order, near $\hat{\omega} = 0$, the jump in $\mathscr{A}$ represents a jump both in the northward flux of total wave energy F and in the northward transfer of zonal momentum. Depending on α and U_0' the jump in F is either positive, in which case the R.w.c.l. will act as an energy emitter, or negative and then the R.w.c.l. will act as an energy absorber. This result will be exploited in § 6 below to study the infludnce of the R.w.c.l. on the reflexion and transmission (as well as on the stability) of planetary waves by a finite shear.

5.2. *Gravity wave critical latitudes*

In the vicinities of the critical latitudes at $\hat{\omega}^2 = N^2$ we expand U about y_c, where y_c is any one of the latitudes

$$U_c = U(y_c) = (\omega \pm N)/k. \tag{5.8}$$

We obtain

$$\psi'' + \left[\frac{-\gamma_1}{4} + \frac{\gamma_2}{y-y_c} + \frac{\frac{1}{4}+\mu^2}{(y-y_c)^2}\right]\psi = 0, \tag{5.9}$$

in which

$$\left.\begin{aligned} \gamma_1 &= g^2k[2kNU_c' \pm (2\beta f - \beta U_c' - fU_c'')]/(4N^3f^2U_c), \\ \gamma_2 &= [2N(fU_c'^2 + \beta U_c') \mp fU_c(kU_c'^2 + NU_c'')]/(2fU_c\,U_c'), \\ \mu^2 &= m^2f^2N^2U_c^2/(4g^2k^2U_c'^2). \end{aligned}\right\} \tag{5.10}$$

If we let

$$\zeta = \gamma_1^{\frac{1}{2}}(y-y_c), \tag{5.11}$$

then (5.9) takes the form

$$\frac{d^2\psi}{d\zeta^2} + \left[-\frac{1}{4} + \frac{\kappa}{\zeta} + \frac{\frac{1}{4}+\mu^2}{\zeta^2}\right]\psi = 0, \tag{5.12}$$

in which

$$\kappa = \gamma_2\gamma_1^{-\frac{1}{2}}. \tag{5.13}$$

Thus $\psi(\zeta)$ satisfies Whittaker's equation the solutions of which are

$$\left.\begin{aligned} M_{\kappa,\,i\mu}(\zeta) &= e^{-\frac{1}{2}\zeta}\zeta^{\frac{1}{2}+i\mu}M(\tfrac{1}{2}+i\mu-\kappa,\,1+2i\mu,\,\zeta), \\ W_{\kappa,\,i\mu}(\zeta) &= e^{-\frac{1}{2}\zeta}\zeta^{\frac{1}{2}+i\mu}U(\tfrac{1}{2}+i\mu-\kappa,\,1+2i\mu,\,\zeta), \end{aligned}\right\} \tag{5.14}$$

where M and U are Kummer's functions defined by (see Abramowitz & Stegun 1965, p. 505)

$$\left.\begin{aligned} M(a,b,\zeta) &= 1 + \frac{a\zeta}{b} + \frac{(a)_2\,\zeta}{(b)_2\,2!} + \dots + \frac{(a)_n\,\zeta^n}{(b)_n\,n!} + \dots, \\ U(a,b,\zeta) &= \frac{\pi}{\sin(\pi b)}\left\{\frac{M(a,b,\zeta)}{\Gamma(1+a-b)\,\Gamma(b)} - \zeta^{1-b}\frac{M(1+a-b,2-b,\zeta)}{\Gamma(a)\,\Gamma(2-b)}\right\} \end{aligned}\right\} \tag{5.15}$$

with

$$(a)_n = a(a+1)\,(a+2)\dots(a+n-1), \quad a_0 = 1. \tag{5.16}$$

Now equation (5.11) is the same as that studied by Jones (1968) for the classical gravity wave critical level. A property of (5.12) is that if $M_{\kappa,\,i\mu}$ and $W_{\kappa,\,i\mu}$ are solutions so are $M_{\kappa,\,-i\mu}$ and $W_{\kappa,\,-i\mu}$. Following Jones (1968); Eltayeb & McKenzie (1975) in their study on the gravity wave reflexion by a shear layer assumed that the two solutions for $\psi(\zeta)$ were $M_{\kappa,\,\pm i\mu}$. However, an investigation of the asymptotic behaviour of $M_{\kappa,\,\pm i\mu}$ shows that they behave like

$$\zeta^{\frac{1}{2}\pm i\mu} \quad \text{for} \quad |\zeta| \to 0.$$

They therefore fail to represent the correct solution for $|\zeta| \to 0$ when $\mu = 0$, because they are not linearly independent, in which case a logarithmic singularity occurs in one of the solutions, because $M(a,b,\zeta) = 1$ for $\zeta = 0$. However, the asymptotic properties of $U(a,b,\zeta)$ in the neighbourhood of $\zeta = 0$ exhibit a logarithmic singularity when $\mu = 0$. Since the solution must be analytically continuous in μ, the two independent solutions of (5.12) must then be chosen as

$M_{\kappa,\,\mathrm{i}\mu}$ and $W_{\kappa,\,\mathrm{i}\mu}$. It should be pointed out, however, that provided $\mu \neq 0$, the behaviour of $M_{\kappa,\,\mathrm{i}\mu}$ and $W_{\kappa,\,\mathrm{i}\mu}$ *in the neighbourhood* of $\zeta = 0$ is similar to that of $M_{\kappa,\,\pm\mathrm{i}\mu}$, and therefore the main conclusions obtained by Eltayeb & McKenzie (1975) for a *thin* shear layer for $\mu \neq 0$ hold good. The numerical results by Jones (1968) may however require some adjustment when $\mu \approx 0$.

Thus the legitimate solution of (5.12) is

$$\psi = A_1 M_{\kappa,\,\mathrm{i}\mu}(\zeta) + A_2 W_{\kappa,\,\mathrm{i}\mu}(\zeta). \tag{5.17}$$

In the neighbourhood of y_c this can be approximated by

$$\psi = \begin{cases} A_1(y-y_c)^{\frac{1}{2}+\mathrm{i}\mu} + B_1(y-y_c)^{\frac{1}{2}-\mathrm{i}\mu} & \text{for} \quad \mu \neq 0, \\ (y-y_c)^{\frac{1}{2}}[\bar{A}_1 + \bar{B}_1 \ln(y-y_c)] & \text{for} \quad \mu = 0. \end{cases} \tag{5.18}$$

The solution is therefore identical to that obtained for the classical gravity wave critical level in which case the correct solutions on either side of the branch point $y = y_c$ is well known (Booker & Bretherton 1967; Baldwin & Roberts 1970). However, the matching of (5.18) is not germane to the analysis below and it is essential to revert to the dependent variable p because the transformations (2.22) and (2.25) are both singular at $\hat{\omega}^2 = N^2$. Now

$$\left. \begin{aligned} h &= \frac{a_1}{(y-y_c)^{\frac{1}{2}}}, \quad a_1 = \frac{N^2 f U}{g|2\hat{\omega}_c k U_c'|^{\frac{1}{2}}}, \\ \chi &= (y-y_c)^{\mathrm{i}\mu}, \end{aligned} \right\} \tag{5.19}$$

so that

$$p = \begin{cases} A(y-y_c)^{2\mathrm{i}\mu} + B & \text{for} \quad \mu \neq 0, \\ A + B\ln(y-y_c) & \text{for} \quad \mu = 0. \end{cases} \tag{5.20}$$

The matching of p across $y - y_c$ can be obtained by simulating an initial value problem in which ω is assumed to have a small negative imaginary part so that at a particular point the disturbance grows with time (Miles 1961). As a result $y - y_c$ in (5.20) is replaced by $y - y_c - \mathrm{i}\omega_\mathrm{i}/(kU_c')$, where $\omega = \omega_\mathrm{r} + \mathrm{i}\omega_\mathrm{i}$. If

$$\theta = \arg(y - y_c - \mathrm{i}\omega_\mathrm{i}/kU_c'), \tag{5.21}$$

then

$$\tan\theta = \frac{-(\omega_\mathrm{i})/kU_c'}{(y-y_c)}. \tag{5.22}$$

As y increases from values much less than y_c to values much greater than y_c, θ increases continuously from 0 to $\mp\pi$ according to whether $kU_c' \gtrless 0$. The correct solutions in the vicinity of y_c can then be written as

$$p = \begin{cases} A(y_c-y)^{2\mathrm{i}\mu} + B & \text{for} \quad y < y_c, \\ \bar{A}(y-y_c)^{2\mathrm{i}\mu} + B & \text{for} \quad y > y_c, \end{cases} \tag{5.23}$$

in which

$$\bar{A} = A\,\mathrm{e}^{\mp 2\pi\mu} \tag{5.24}$$

for $\mu \neq 0$; and

$$p = \begin{cases} A + B\ln(y_c - y) & \text{for} \quad y < y_c, \\ A + B[\ln(y-y_c) \mp \mathrm{i}\pi] & \text{for} \quad y > y_c, \end{cases} \tag{5.25}$$

when $\mu = 0$.

Now we must express $\mathscr{A}$ in terms of the pressure p:

$$\mathscr{A} = \frac{1}{|\chi|^2|h|^2}\left\{\operatorname{Im}(p^* p') - |p|^2 \operatorname{Im}\left(\frac{h'}{h} + \frac{\chi'}{\chi}\right)\right\}. \tag{5.26}$$

Straightforward calculations then yield

$$\left.\begin{aligned} \mathscr{A} &= \frac{\mu}{|a_1|^2}(|A|^2-|B|^2) \quad \text{for} \quad y < y_c, \\ \mathscr{A} &= \frac{\mu}{|a_1|^2}[|A|^2\exp(\mp 4\pi\mu)-|B|^2] \quad \text{for} \quad y > y_c, \end{aligned}\right\} \tag{5.27}$$

when m is real;

$$\left.\begin{aligned} \mathscr{A} &= \frac{2\nu}{|a_1|^2}\mathrm{Im}\,(A^*B) \quad \text{for} \quad y < y_c, \\ \mathscr{A} &= \frac{2\nu}{|a_1|^2}\mathrm{Im}\,[A^*B\exp(\pm 2\pi\nu i)] \quad \text{for} \quad y > y_c, \end{aligned}\right\} \tag{5.28}$$

in which

$$\nu^2 = -\mu^2 \quad (>0), \tag{5.29}$$

for imaginary m; and

$$\left.\begin{aligned} \mathscr{A} &= \frac{1}{|a_1|^2}\mathrm{Im}\,(A^*B) \quad \text{for} \quad y < y_c, \\ \mathscr{A} &= \frac{1}{|a_1|^2}\mathrm{Im}\,(A^*B) \pm \frac{\pi|B|^2}{|a_1|^2} \quad \text{for} \quad y > y_c, \end{aligned}\right\} \tag{5.30}$$

when $m = 0$. Note that the upper (lower) signs refer to $kU_c' > (<) 0$.

The expressions (5.27)–(5.30) show the strong influence of m on the discontinuity in $\mathscr{A}$ at the g.w.c.l. For real m the solution (B) is not affected by the presence of the critical latitude and therefore represents the wave permitted by the valve behaviour to propagate across that latitude while the (A) solution is attenuated across the same latitude. However, the invariant $\mathscr{A}$ is composed of both solutions and it is not legitimate to view the two independent solutions A and B individually in terms of $\mathscr{A}$. Thus the conclusion to be reached from (5.27)–(5.30) is that the wave invariant $\mathscr{A}$ is discontinuous there. In order to determine whether energy is absorbed at $\hat{\omega} = \pm N$ or not we require additional information. Suppose m is real and non-zero and consider the critical latitude at $\hat{\omega} = -N$. By appealing to figures 3 and 4 we see that kU_c' is positive or negative if, respectively, the transmitted wave (by the valve-behaviour) has a positive or negative group velocity in the y-direction (where positive and negative group velocity is away from, or towards, the observer). If we realize that the B solution is the transmitted wave and therefore the A solution has a group velocity opposite to that of the B solution and hence has the same sign as kU_c', wee see that the negative sign in (5.27) is relevant. The same result holds for the other critical latitude at $\hat{\omega} = N$. We then conclude that $\mathscr{A}$ is *always reduced* across the g.w.c.l. if m is real and non-zero.

To interpret the discontinuity of $\mathscr{A}$ in terms of energy flux and momentum transfer in the northward direction we appeal to the expressions (4.3) and (4.5). For real values of m, $|\chi| = 1$ and F and M are (for a particular wave) proportional to $\mathscr{A}$ and hence both F and M are reduced across the g.w.c.l. If m is imaginary, however, then

$$|\chi|^2 = |y-y_c|^{2b}, \quad b = (\mathrm{i}m)fU_c\,N^2/g2\hat{\omega}_c\,kU_c', \tag{5.31}$$

and the energy flux and momentum transfer either approach zero or tend to infinity depending on whether b is positive or negative at the g.w.c.l. concerned.

The g.w.c.l. in a rotating fluid is then different from the classical gravity wave critical level. In addition to the valve behaviour we shall see in § 6 below that the present g.w.c.l. has a profound effect on the reflectivity and stability of atmospheric waves.

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5.3. *Gravity-inertial wave singularities*

In the neighbourhood of $Q = 0$ the equation (2.26) can be approximated by

$$\psi'' - \frac{\frac{3}{4}}{(y-y_1)^2}\psi = 0, \tag{5.32}$$

where y_1 stands for any of the four singularities occurring where $Q = 0$. The solution of (5.32) which can be matched directly across y_1 because both χ and h are continuous there is

$$\psi = \begin{cases} A_2(y_1-y)^{\frac{3}{2}} + B_2(y_1-y)^{-\frac{1}{2}} & \text{for} \quad y < y_1, \\ \pm \mathrm{i}A_2(y-y_1)^{\frac{3}{2}} \pm \mathrm{i}B_2(y-y_1)^{-\frac{1}{2}} & \text{for} \quad y > y_1, \end{cases} \tag{5.33}$$

where the upper (lower) sign refers to $kU'(y_1) > (<0)$. Direct evaluation of $\mathscr{A}$ using (4.1) shows that

$$\mathscr{A} = 2\,\mathrm{Im}\,(A_2 B_2^*)$$

on both sides of y_1 and hence $\mathscr{A}$ is continuous across every singularity occurring where $Q = 0$.

6. Reflexion of Rossby-gravity waves by a finite shear layer

The study of reflexion of planetary waves by a vortex sheet in M.M. indicated that over-reflexion is possible only if the waves are evanescent in the vertical direction so that they can propagate eastward in phase for small values of the flow. When these conditions are satisfied the over-reflecting régimes occur when a Rossby critical latitude is embedded inside the vortex sheet. The vortex sheet treatment naturally neglects the presence of the critical latitude there and taking into account the detailed analysis carried out by Eltayeb & McKenzie (1975) for the gravity wave critical level in a Boussinesq (non-rotating) fluid one might expect, as is claimed in M.M. that the results of the vortex sheet may provide a good approximation to the case of a thin shear layer (in the sense that the wavelength of the oncoming wave is much longer than the thickness of the shear, i.e. $|kL| \ll 1$). However, recent studies on other critical levels (El Sawi & Eltayeb 1978; Eltayeb 1980) showed that whether the vortex sheet treatment will tally with the thin shear layer or not depends very much on what type of critical level (or latitude) is relevant. Even in the absence of critical levels the stability of a finite shear is known to be different from that of the vortex sheet (see Blumen *et al.* 1975; Eltayeb 1980). It is therefore necessary to examine the influence of the Rossby and gravity wave critical latitudes in the present context on the reflectivity and transmissivity of the waves as well as on their stability.

Consider a finite shear layer of thickness L in an unbounded medium, as defined in § 2 above, and suppose that the flow distribution is given by

$$U = \begin{cases} U_1, & y \leqslant 0 \quad \text{(region I)}, \\ U(y), & 0 \leqslant y \leqslant L \quad \text{(region II)}, \\ U_3, & L \leqslant y \quad \text{(region III)}, \end{cases} \tag{6.1}$$

in which it is assumed that $U(y)$ is continuous at both $y = 0, L$. The boundary conditions applicable at $y = 0, L$ are the continuity of both pressure p_1 and the northward component of velocity v_1.

$$\langle p_1 \rangle = 0, \quad \langle v_1 \rangle = 0 \quad \text{at} \quad y = 0, L, \tag{6.2}$$

where the angle brackets denote the jump in the quantity within. By using (2.20) equation (6.2) can be expressed in terms of the field variable p

$$\langle p\rangle = 0, \quad \langle Ep/Q+\hat{\omega}(N^2-\hat{\omega}^2)\,p'/Q\rangle = 0 \quad \text{at} \quad y = 0, L. \tag{6.3}$$

Another form of the boundary conditions, which will be useful below, is

$$\langle h\psi\rangle = 0 \quad \text{at} \quad y = 0, L \tag{6.4}$$

$$\langle\{E+\hat{\omega}(N^2-\hat{\omega}^2)\,(h'/h+\chi'/\chi)\}\,h\psi/Q\rangle+\hat{\omega}(N^2-\hat{\omega}^2)\,\langle h\psi/Q\rangle = 0 \quad \text{at} \quad y = 0, L, \tag{6.5}$$

where we have used the continuity of U at $y = 0, L$.

Consider a wave, amplitude I, incident on the shear layer (region II) from region I. It will give rise to a reflected wave, amplitude $|R|$, in region I and a transmitted wave, amplitude $|T|$, in region III. The solutions of (2.26) can then be written

$$\psi = \begin{cases} I\exp(\mathrm{i}n_1 y)+R\exp(-\mathrm{i}n_1 y) & \text{(region I)},\\ A_1\psi_1(y)+A_2\psi_2(y) & \text{(region II)},\\ T\exp(\mathrm{i}n_3 y) & \text{(region III)}, \end{cases} \tag{6.6}$$

where ψ_1 and ψ_2 are the two linearly independent solutions of (2.26) in region II, and n_1, n_3 are given by

$$n_i^2 = -k^2-(m^2+\tfrac{1}{4}H^{-2})\,\frac{(\hat{\omega}_i^2-f^2)}{(N^2-\hat{\omega}_i^2)}-k\beta(\hat{\omega}_i^2-N^2)\,\frac{(\hat{\omega}_i^2+f^2)}{\hat{\omega}_i\,Q_i}$$
$$+\beta^2(\hat{\omega}_i^2-N^2)^2\,\frac{(\hat{\omega}_i^2+2f^2)}{Q_i^2}+\frac{m^2f^2U_i^2\,N^4}{g^2(\hat{\omega}_i^2-N^2)^2}, \tag{6.7}$$

in which

$$Q_i = (N^2-\hat{\omega}_i^2)\,(\hat{\omega}_i^2-f^2)+\hat{\omega}_i^2f^2U_i^2\,N^2/g^2, \quad \hat{\omega}_i = \omega-kU_i \quad \text{for} \quad i = 1, 3. \tag{6.8}$$

Here n_1 and n_3 must be chosen in such a way that the incident wave transports energy *towards* the layer while the reflected and transmitted waves transport energy *away* from the layer if both n_1 and n_3 are real. If n_3 is imaginary then it must be chosen such that $\mathrm{i}n_3 < 0$. n_1 is, by hypothesis, real.

6.1. *General considerations*

Here we shall employ the wave-invariant $\mathscr{A}$ to deduce some general results for the reflexion by a shear layer of arbitrary (finite) thickness, L, when U' and U'' are both continuous everywhere so that discontinuities in $\mathscr{A}$ can only be due to critical latitudes.

In region I the invariant, $\mathscr{A}_1$, is given by

$$\mathscr{A}_1 = n_1(|I|^2-|R|^2), \tag{6.9}$$

while in region III it takes the form

$$\mathscr{A}_3 = \mathrm{Re}\,(n_3)\,|T|^2. \tag{6.10}$$

In the absence of critical latitudes within the shear $\mathscr{A}_1 = \mathscr{A}_2$ and hence

$$|R|^2 = |I|^2-[\mathrm{Re}\,(n_3)/n_1]\,|T|^2. \tag{6.11}$$

Thus over-reflexion can occur only if

$$\mathrm{Re}\,(n_3)/n_1 < 0. \tag{6.12}$$

If n_3 is imaginary, and the transmitted wave is evanescent in the northward direction, perfect reflexion occurs. For real n_3 over-reflexion occurs for $n_1 n_3 < 0$. To identify the over-reflecting régimes we inspect the wave normal curves in regions I and III (cf. figures 1–4). By using the notation of figure 15 and employing a superscript to denote the region, i.e. $k_1^{(1)}$ refers to k_1 in region I, the over-reflecting régimes can be located as follows:

(i) $\omega^2 < f^2$ *and m real*

In a westerly flow increasing with latitude waves with eastward and westward phase propagation can be over-reflected. (i*a*) If a wave is westward propagating and provided U_1 is small enough to make $k_\pm$ real and U_3 is such that $k_3^{(3)} < k_+^{(1)}$ then it will be over-reflected if k lies in the range

$$\max [k_-^{(1)}, (\omega - N)/U_3] < k < \min [k_{3+}^{(3)}, k_+^{(1)}]. \tag{6.13}$$

(i*b*) If a wave is eastward propagating and, provided $U_1 \neq 0$, it will be over-reflected if k satisfies

$$\max [k_{4-}^{(3)}, \omega/U_1] < k < \min [k_1^{(1)}, (\omega + N)/U_3]. \tag{6.14}$$

It is noteworthy that the over-reflecting régime (i*a*) applies to the case studied in M.M. when m is real. The fact that M.M. failed to predict it is because the assumption $\hat{\omega}^2 \ll N^2$ made in M.M. filtered out the possible transmitted (gravity) wave. It is also to be noted that in both (i*a*) and (i*b*) the incident wave is a Rossby wave (for which $\hat{\omega}^2 \ll N^2$) while the transmitted wave is a gravity wave (with $\hat{\omega}^2 \approx N^2$). However, the régimes (i*a*) and (i*b*) are different in the sense that while (i*a*) occurs only when the incident (and reflected) wave has a long wavelength the régime (i*b*) can occur for both long and short wavelengths provided U_1 and U_3 are such that $k_1^{(1)} > k_{4-}^{(3)}$.

(ii) $f^2 < \omega^2 < N^2$ *and m real*

(ii*a*) In a westerly flow increasing in latitude a wave with eastward phase propagation can be over-reflected. The over-reflecting régime here is given by (6.14).

(ii*b*) In an easterly flow increasing with latitude only westward propagating (in phase) waves can be over-reflected. Provided $|U_3|$ is large enough to make $k_1^{(3)} > k_-^{(1)}$ and provided $s^2 > 0$ (cf. (3.12)) then all waves with k in the range

$$\max [k_-^{(1)}, \omega/U_3] < k < k_1^{(3)}, \tag{6.15}$$

are over-reflected.

When one or more critical latitudes exist within the shear, $\mathscr{A}$ will be discontinuous at every one of them. We shall now use the relations obtained for $\mathscr{A}$ near each critical latitude (cf. § 5) to draw some general conclusions regarding R and T.

If a Rossby wave critical latitude exists within the shear and no other critical latitudes are present then we can use the relation (5.6) together with (6.9) and (6.10) to obtain

$$|R|^2 = |I|^2 - |T|^2 \operatorname{Re}(n_3)/n_1 + \pi\bar{\alpha}|A|^2 \operatorname{sgn}(k)/n_1|U_0'|, \tag{6.16}$$

where

$$\bar{\alpha} = \alpha U_0' \tag{6.17}$$

(cf. (5.4)). Thus the presence of a R.w.c.l. will enhance over-reflexion if $n_1 \bar{\alpha} \operatorname{sgn}(k) > 0$, otherwise it will tend to oppose it. This indicates that the logarithmic derivative of the potential vorticity (which equals $f\bar{\alpha}$) at the critical latitude has a strong influence on the reflexion of Rossby waves by a finite shear (see Dickinson & Clare 1973).

In the situations when a critical latitude at $\hat{\omega}^2 = N^2$ lies within the shear no simple relation between R and T is obtainable for the general case. However, such situations will be studied in detail below. The case when the limit L tends to zero is considered in the next subsection.

The above relations between the reflexion and transmission coefficients for smoothly varying shear flows (in the sense that both U and its first two derivatives are continuous everywhere) can be used to deduce general results about the relations between over-reflecting and unstable régimes. In the absence of critical latitudes the unstable modes must obey the relation

$$\mathrm{Re}\,(n_1)\,|R|^2+\mathrm{Re}\,(n_3)\,|T|^2 = 0, \tag{6.18}$$

so that if n_1 is real (imaginary) n_3 must also be real (imaginary). Although the system may possess unstable modes for which n_1 is imaginary the situation relevant to over-reflexion is associated with real n_1. Thus any mode of instability for which *both* n_1 and n_3 are real will yield an infinite $|R/I|$, according to (2.11), i.e. resonant over-reflexion will occur. In this case if the unstable mode is associated with a given set of the parameters (i.e. wave numbers, flow speeds, etc.) then slight departure from these given values will give over-reflexion. Thus all unstable modes associated with n_1, n_3 real are accompanied by resonant over-reflexion. It should be remarked here that these deductions apply to all types of smooth shear (of the velocity and/or magnetic type) provided the conditions for the existence and continuity of the wave-invariant are met.

However, the relations obtained above cannot provide information about whether over-reflexion is possible in a *stable* shear. To clarify this interesting situation we shall study the relatively simple case of a shear the thickness of which is much smaller than the zonal wavelength of the incident wave (i.e. when $kL \ll 1$). Now for a flow $U(y)$ to vary appreciably over the small thickness L of the layer (assuming $k = O(1)$ it must depend on the scaled variable $Y(=y/L)$, and in general can be written in the form

$$U(y) = U_1+U_0\,F(Y), \tag{6.19}$$

in which
$$U_0 = U_3-U_1, \quad F(0) = 0, \quad F(1) = 1. \tag{6.20}$$

It then follows that $\partial U/\partial y = O(L^{-1})$ and hence it increases indefinitely as L approaches zero. The profile (6.19) may then resemble a vortex sheet in this limit of vanishing layer thickness and it is interesting to see how the results of the thin shear compare with those of the corresponding vortex sheet. Even in this limit the general problem requires extensive numerical work and we will therefore limit ourselves to the case of a linear shear in which case analytical results are obtained for the full range of $\hat{\omega}/N$. The influence of the R.w.c.l. is also found to depend on the type of the profile $F(Y)$ and to illustrate this we briefly discuss the general smooth profile in §6.3 below.

6.2. *The linear shear*

Here $F(Y) = Y$ and consequently the gradient of F is unity within the layer $0 \leqslant Y \leqslant 1$. The system (2.26), (2.27) and (6.5) reduces to

$$\frac{\mathrm{d}^2\overline{\psi}}{\mathrm{d}X^2}+\left[\frac{\alpha}{XS}+\frac{1+4\mu^2}{(X^2-1)^2}+\frac{1-2X^2}{(X^2-1)^2S^2}+O(L^2)\right]\overline{\psi} = 0, \tag{6.21}$$

$$\left.\begin{aligned}
&\gamma_1\psi(0-) = \delta\overline{\psi}(0+),\\
&\gamma_3\psi(1+) = \delta\overline{\psi}(1-),\\
&\gamma_1^{-1}\{-k\delta\psi(0-)+X_1\psi'(0-)\} = kX_1\overline{\psi}(0+),\\
&\gamma_3^{-1}\{-k\delta\psi(1+)+X_3\psi'(1+)\} = kX_3\overline{\psi}(1-),
\end{aligned}\right\} \tag{6.22}$$

where
$$X = \hat{\omega}/N, \quad \delta = f/N, \quad \overline{\psi} = (-fU_0/L)^{\frac{1}{2}}\psi$$
$$\gamma_1 = (\delta^2-X_1^2)^{\frac{1}{2}}, \quad \gamma_3 = (\delta^2-X_3^2)^{\frac{1}{2}},$$

$$\alpha = \beta L^2 N/(kU_0^2), \quad \tau = f^2U^2/g^2,$$
$$S = 1 + (LN^2/fU_0)\{X^2 - \tau X^2/(X^2-1)\}, \tag{6.23}$$

and the terms $O(L^2)$ in (6.21) are nonsingular for any value of X.

The formulation of the problem in terms of the scaled frequency X simplifies the problem considerably by reducing the number of parameters of the original problem because only the ratios of the different frequencies are now relevant. Since the problem spans a wide variety of frequencies ranging from high frequency gravity waves ($X \approx 1$) to low frequency planetary waves ($X \ll \delta \ll 1$) we find it convenient to study the following three cases separately.

(i) $|X| \ll 1$

This case includes the planetary waves discussed in M.M. The governing equation (6.21) takes the simple form

$$\frac{d^2\overline{\psi}}{dX^2} + \left(2 + \frac{\alpha}{X}\right)\overline{\psi} = 0 \tag{6.24}$$

This equation immediately shows that the propagation properties of planetary waves using a full wave treatment are very different from those in the W.K.B.J. treatment. Except in the region $|X| \leqslant O(|\alpha|)$ the solution of (6.24) is oscillatory. Assuming that $\hat{\omega}_1 > 0$ we see that (since $\alpha = O(L^2)$ here) the solution is oscillatory within the shear except in a very thin region in the neighbourhood of the R.w.c.l. Moreover, since α is positive (negative) for westerly (easterly) flows if a R.w.c.l. lies within the shear then the region between $k = 0$ and $k = k_\infty$ in figure 1 is a propagation region and no evanescence occurs there. For the easterly flow (figure 2) on the other hand propagation occurs in the region $0 > k > K$ where K is slightly greater than k_∞ and evanescence occurs in the small interval $K \geqslant k > k_\infty$ the thickness of which is $O(|\alpha|)$. However, more interestingly, propagation in this last case also occurs in the region $k < k_\infty$ since both α and X are negative there. Consequently the reflexion and transmission properties of the thin shear can be expected to be very different from those of the vortex sheet. This will be made clear below.

To illustrate the role of the R.w.c.l. we will study the two subcases (i*a*) shear free of critical latitudes and (i*b*) shear containing one R.w.c.l. separately.

(i*a*) *Shear free of critical latitudes*

By assuming that $|X| \gg |\alpha|$ the two independent solutions of (6.24) are

$$\overline{\psi}_1 = \exp(iX\surd 2), \quad \overline{\psi}_2 = \exp(-iX\surd 2), \tag{6.25}$$

which represent two propagating waves one northward going and the other southward-going. The use of (6.22) and (6.6) yields

$$\frac{R}{I} = \frac{A - iB}{C + iD}, \quad \frac{T}{I} = \left(1 + \frac{R}{I}\right)\frac{\delta_1 \exp(iX_3\surd 2) + \exp(-iX_3\surd 2)}{\delta_1 \exp(iX_1\surd 2) + \exp(-iX_1\surd 2)}. \tag{6.26}$$

where

$$\delta_1 = \frac{A_1}{A_2} = \left\{\frac{1 - iX_3\surd 2 - i\bar{n}_3 X_3/\delta}{-1 - iX_3\surd 2 + i\bar{n}_3 X_3/\delta}\right\}\exp(-2iX_3\surd 2),$$
$$A = 1 - (\bar{n}_1 + \delta\surd 2)(\bar{n}_3 - \delta\surd 2)X_1X_3/\delta^2,$$
$$B = (\bar{n}_1 X_1 + \bar{n}_3 X_3)/\delta + \sigma_1,$$
$$C = -1 - \bar{n}_1\bar{n}_3 X_1 X_3/\delta^2 - 2X_1X_3 - \sigma_1 \cot\sigma_1,$$
$$D = (\bar{n}_1 X_1 + \bar{n}_3 X_3)/\delta + \delta^{-1}(\bar{n}_1 + \bar{n}_3)X_1X_3\surd 2 \cot\sigma_1, \tag{6.27}$$

in which $$\sigma_1 = \surd 2\,(X_1 - X_3), \quad \bar{n}_{1,3} = n_{1,3}/k. \tag{6.28}$$

If both X_1 and X_3 are small compared to δ (i.e. the case studied in M.M. for the vortex sheet) then

$$\frac{R}{I} \approx \frac{\delta \tan \sigma_1}{\mathrm{i}\surd 2\,(\bar{n}_1 X_1 + \bar{n}_3 X_3)}, \tag{6.29}$$

so that $$\left|\frac{R}{I}\right|^2 \approx \tfrac{1}{2}\delta^2 \tan^2 \sigma_1 \,|\bar{n}_1 X_1 + \bar{n}_3 X_3|^{-2} \ll 1, \tag{6.30}$$

since $\delta^2 \ll 1$ and $\bar{n}_1$ and $\bar{n}_3$ have the same sign if both are real. If, on the other hand, $X_{1,3} = O(\delta)$ then

$$\left|\frac{R}{I}\right|^2 \approx \tfrac{1}{2}\tan^2 \sigma_1 \left[1 + \left(\frac{\delta^2 - \bar{n}_1 \bar{n}_3 X_1 X_3}{\bar{n}_1 X_1 + \bar{n}_3 X_3}\right)^2\right] \ll 1 \tag{6.31}$$

for real $\bar{n}_3$; and $$\left|\frac{R}{I}\right|^2 \approx \tfrac{1}{2}\tan^2 \sigma_1 \left[\frac{(\delta^2 + \bar{n} X_3)^2\,(1 + \bar{n}_1^2 X_1^2/\delta^2)}{\bar{n}_1^2 X_1^2 + \bar{n}^2 X_3^2}\right] \ll 1, \tag{6.32}$$

for imaginary $\bar{n}_3 (= \mathrm{i}\bar{n}, \bar{n} > 0)$. In all these cases the reflexion coefficient is directly proportional to the jump ($|U_0|$) in the flow speed across the layer and since this, by hypothesis, is small no over-reflexion is possible. Furthermore in contrast to the vortex sheet treatment no perfect reflexion is possible even when n_3 is imaginary the reason being that the wave propagates all the way across the shear layer and becomes evanescent only in region III. In this way energy is transmitted to the near end of region III although it decays very rapidly as y increases from L.

(i*b*) *Shear containing an R.w.c.l.*

When $X = 0$ lies within the layer (6.24) must be solved in its entirety. It transpires, however, that it can be transformed into Whittaker's equation and the two solutions are simply

$$\bar{\psi}_1 = \mathrm{e}^{-\frac{1}{2}Z} Z M(1-\kappa, 2, Z), \quad \bar{\psi}_2 = \mathrm{e}^{-\frac{1}{2}Z} Z\, U(1-\kappa, 2, Z), \tag{6.33}$$

where $$Z = 2\mathrm{i}X\surd 2, \quad \kappa = \mathrm{i}\alpha/(16\surd 2) \tag{6.34}$$

and M and U are Kummer's functions defined in (5.15) above. Thus for small values of Z appropriate here we have

$$\begin{aligned} U(1-\kappa, 2, Z) = \frac{1}{2\Gamma(-\kappa)} \Big\{ & M(1-\kappa, 2, Z) \ln Z \\ & + \sum_{r=0}^{\infty} \frac{(1-\kappa)_r Z^r}{(2)_r\, r!} [\phi(1-\kappa+r) - \phi(1+r) - \phi(2+r)] \Big\} \\ & + \frac{1}{\Gamma(1-\kappa)\, Z} M(1-\kappa, 0, Z), \end{aligned} \tag{6.35}$$

in which Γ is the usual gamma function and

$$\phi(a) = \Gamma'(a)/\Gamma(a), \tag{6.36}$$

the accent denoting differentiation with respect to the argument. The notations $M(a, b, Z)$ and $(x)_r$ are defined in (5.15) and (5.16) above. It should be pointed out here that an investigation of the asymptotic behaviour of the functions M and U for $|X| > |\alpha|$ matches uniformly with the solution (6.25), as would be expected.

The application of the boundary conditions (6.22) to the solution (6.33) and (6.6), omitting the lengthy but straightforward manipulations, yields the following expressions for the reflexion and transmission coefficients

$$\frac{R}{I}=\frac{A+\mathrm{i}B}{C+\mathrm{i}D},\quad \frac{T}{I}=a\left(1+\frac{R}{I}\right), \tag{6.37}$$

where

$$\left.\begin{aligned}
a &= \bar{\gamma}_1\bar{\gamma}_3\{\gamma(1+2\bar{\gamma}_3)+(\mathrm{i}\bar{n}_3\bar{X}_3-1)\ln(X_1/X_3)\}^{-1},\\
A &= (1+\bar{\gamma}_1^2)(1+\bar{\gamma}_3^2)\ln|X_1/X_3|+\bar{X}_3-\bar{X}_1^2-\epsilon\pi\{\bar{n}_1(1+\bar{\gamma}_3^2)\bar{X}_1+\bar{n}_3(1+\bar{\gamma}_1^2)\bar{X}_3\},\\
B &= -\{\bar{n}_1(1+\bar{\gamma}_3^2)\bar{X}_1+\bar{n}_3(1+\bar{\gamma}_1^2)\bar{X}_3\}\ln|X_1/X_3|+\{\bar{n}_1\bar{\gamma}_3^2-(1+\bar{\gamma}_3^2)(1+2\bar{\gamma}_1^2)/\surd 2\}\bar{X}_1\\
&\quad+\{-\bar{n}_3\bar{\gamma}_1^2+(1+\bar{\gamma}_1^2)(1+2\bar{\gamma}_3^2)/\surd 2\}X_3-\epsilon\pi(1+\bar{\gamma}_1^2)(1+\bar{\gamma}_3^2),\\
C &= -(1+\bar{\gamma}_1^2)(1+\bar{\gamma}_3^2)\ln|X_1/X_3|-2\bar{\gamma}_1^2\bar{\gamma}_3^2-\bar{\gamma}_1^2-\bar{\gamma}_3^2-\epsilon\pi\{\bar{n}_1(1+\bar{\gamma}_3^2)\bar{X}_1-\bar{n}_3(1+\bar{\gamma}_1^2)\bar{X}_3\},\\
D &= -\{\bar{n}_1(1+\bar{\gamma}_3^2)\bar{X}_1-\bar{n}_3(1+\bar{\gamma}_1^2)\bar{X}_3\}\ln|X_1/X_3|+\{\bar{n}_1\bar{\gamma}_3^2+(1+\bar{\gamma}_3^2)(1+2\bar{\gamma}_1^2)/\surd 2\}\bar{X}_1\\
&\quad-\{\bar{n}_3\bar{\gamma}_1^2+(1+\bar{\gamma}_1^2)(1+2\bar{\gamma}_3^2)/\surd 2\}\bar{X}_3+\epsilon\pi(1+\bar{\gamma}_1^2)(1+\bar{\gamma}_3^2),
\end{aligned}\right\} \tag{6.38}$$

in which
$$\bar{\gamma}_{1,3}=\gamma_{1,3}/\delta,\quad \bar{X}_{13}=X_{1,3}/\delta \tag{6.39}$$

and γ is Euler's constant
$$\gamma\approx 0.57721. \tag{6.40}$$

The quantity ϵ is a measure of the phase jump at the critical latitude and takes the values 1, -1 if $kU_c'\gtrless 0$ respectively (assuming that $X_1>0$ and $X_3<0$).

If $|X_3|<|\alpha|$ so that U_3 is close to U_c then $|X_1/X_3|\gg 1$ and the expressions (6.37) can be simplified considerably to become

$$\left|\frac{R}{I}\right|^2\approx 1-\frac{2(1+\bar{\gamma}_1^2)}{[(1+\bar{\gamma}_1^2)^2+\bar{n}_1^2\bar{X}_1^2]\ln|X_1/X_3|} \tag{6.41}$$

$$\left|\frac{T}{I}\right|^2\approx\frac{\bar{\gamma}_1^2}{\{\ln|X_1/X_3|\}^4}\,\frac{(1+\bar{\gamma}_1^2)^4+\bar{n}_1^2\bar{X}_1^2(1+3\bar{\gamma}_1^2)}{[(1+\bar{\gamma}^2)^2+\bar{n}_1^2\bar{X}_1^2]^2}. \tag{6.42}$$

Thus the wave is almost perfectly reflected, although $|R/I|$ is always slightly less than unity. The transmitted wave has a very small amplitude and propagates almost zonally (because $|n_3|\gg|k|$). For larger values of $|X_3|$ the expressions (6.37) must be calculated numerically. A sample of the results is given in figure 16. For comparison reasons the reflexion and transmission coefficients (as well as the stability) of the corresponding vortex sheet (see equations (36)–(39) in M.M.) are also studied in detail numerically and a comparison example is included in figure 17. It is found that both the linear shear and the vortex sheet are stable to all linear disturbances of the normal mode type. However, the reflexion and transmission coefficients are drastically different in the two situations. This is due to two factors. First the presence of the critical latitude can lead to energy emission. However, because of the small value of α for the linear shear (but see § 6.3 below) energy emission is superceded by the second factor due to the strong interaction between the wave and mean flow. Of course both these mechanisms are absent in the vortex sheet treatment. Consequently over-reflexion in the case of the vortex sheet is extremely weak compared with that due to the thin shear even when the parameters in regions I and III are the same in both cases. A typical comparison is made in figure 17.

Before we conclude this case we may note that we have used the notation

$$\bar{\beta}=\beta/kf,\quad \chi=\delta^2\{m^2+(4H^2)^{-1}\}/k^2$$

in figures 16 and 17.

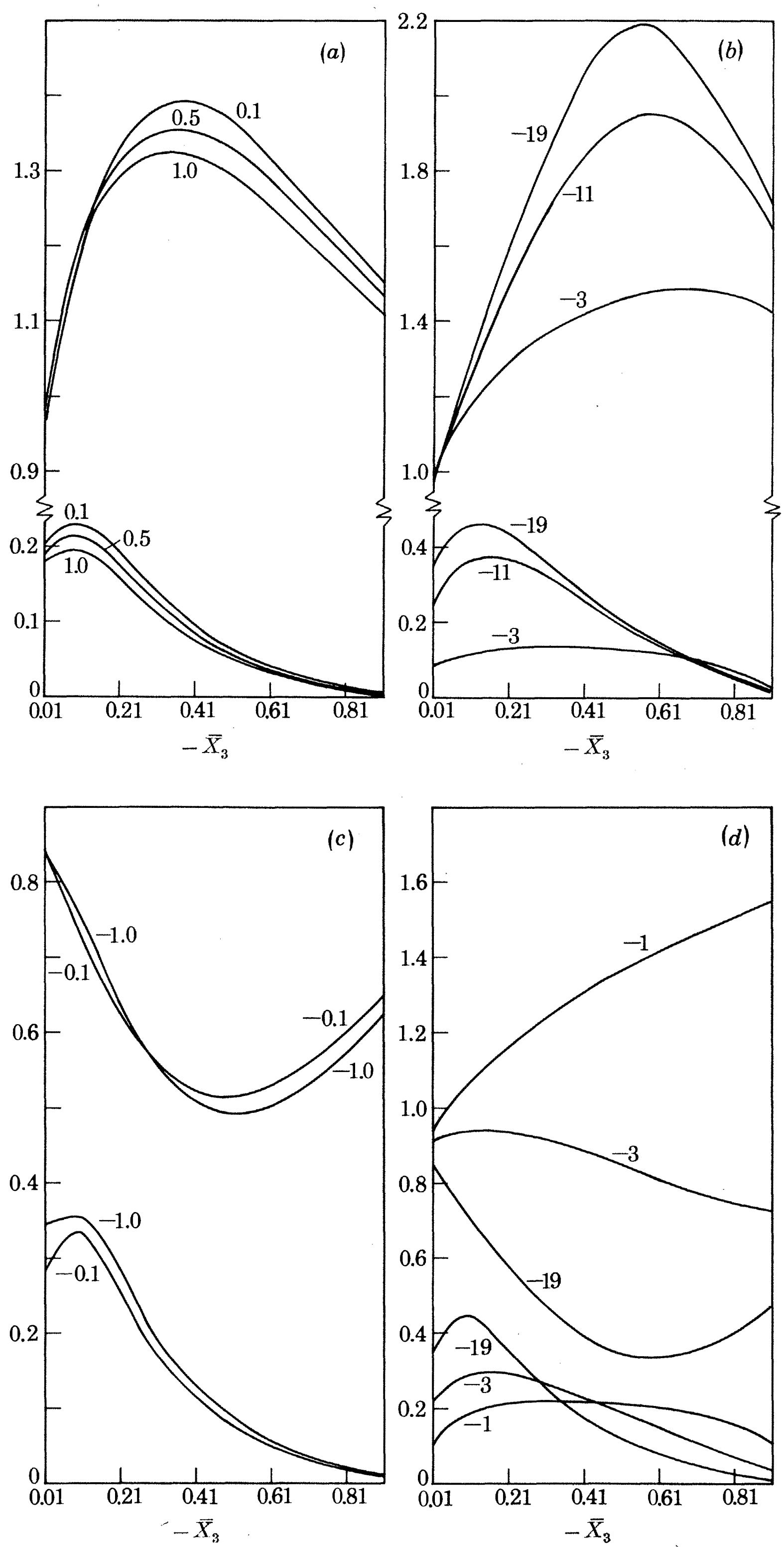

FIGURE 16 Some representative samples of the reflexion and transmission coefficients $|R/I|$ and $|T/I|$ for planetary waves when a R.w.c.l. lies within the shear (see case (i*b*) of § 6.2) drawn as functions of $-\bar{X}_3$ for various values of $\bar{\beta}$ and χ. Curves starting with values below 0.4 on the right correspond to $|T/I|$ and the others to $|R/I|$. (*a*) Westerly flow for which $\bar{X}_1 = 0.25$, $\chi = -19.0$ and $\bar{\beta} = 0.1, 0.5, 1.0$ as marked. (*b*) Westerly flow for which $\bar{X}_1 = 0.5$, $\bar{\beta} = 0.1$ and $\chi = -19.0, -11.0, -3.0$ as marked. (*c*) Easterly flow for which $\bar{X}_1 = 0.25$, $\chi = -19.0$ and $\bar{\beta} = -0.1, -1.0$. (*d*) Easterly flow for which $\bar{X}_1 = 0.5$, $\bar{\beta} = 0.1$ and $\chi = -19.0, -3.0, -1.0$.

(ii) $O(\delta) \leqslant |X| < 1$

In this range of X (6.21) assumes the form

$$\frac{d^2\overline{\psi}}{dX^2}+\frac{2\overline{\psi}}{(1-X^2)} = 0, \tag{6.43}$$

and upon making the transformation

$$\zeta = 1-X^2, \tag{6.44}$$

we find that it becomes

$$\zeta(1-\zeta)\frac{d^2\overline{\psi}}{d\zeta}-\tfrac{1}{2}\zeta\frac{d\overline{\psi}}{d\zeta}+\tfrac{1}{2}\overline{\psi} = 0. \tag{6.45}$$

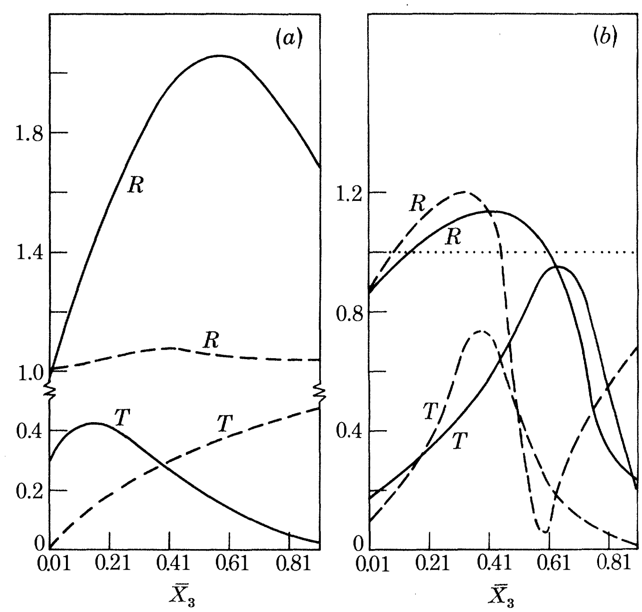

FIGURE 17. Comparison of the results for the shear and the corresponding vortex sheet for planetary waves when a critical latitude lies within the shear. (*a*) The reflexion and transmission coefficients for the shear (continuous curves) and the vortex sheet (discontinuous curves) against $-\overline{X}_3$ for $\overline{\beta} = 0.1$, $\chi = -15.0$ and $X_1 = 0.5$ in a westerly increasing with latitude. (*b*) The reflexion coefficients (R) and transmission coefficients (T) in an easterly when $\overline{X}_1 = 0.1, \overline{\beta} = -1.0$ for $\chi = 1.0$ (continuous curves) and $\chi = 9$ (discontinuous ones). The vortex sheet results are also included: ---, the reflexion coefficient of unity; $T = 0$.

This is the hypergeometric equation and the two independent solutions can be taken as

$$\left.\begin{aligned}\overline{\psi}_1 &= \zeta F(1+a, 1+b;\ 2;\ \zeta),\\ \overline{\psi}_2 &= F(a, b;\ \tfrac{1}{2};\ 1-\zeta),\end{aligned}\right\} \tag{6.46}$$

in which

$$a = -\tfrac{1}{4}(1-\surd 5), \quad b = -\tfrac{1}{4}(1+\surd 5), \tag{6.47}$$

and

$$F(a, b;\ c;\ x) = \sum_{n=0}^{\infty} (a)_n\,(b)_n\,x^n/\{(c)_n\,n!\}, \tag{6.48}$$

where the notation $(a)_n$, etc., is defined in (5.16) above.

Now straightforward manipulations yield

$$\frac{R}{I} = \frac{[(\mathrm{i}\bar{n}_1 \bar{X}_1 - 1)\,\bar{\psi}_1(X_1) - \bar{\gamma}_1^2 X_1 \bar{\psi}_1'(X_1)]\,\delta_1 + (\mathrm{i}\bar{n}_1 \bar{X}_1 - 1)\,\bar{\psi}_2(X_1) - \bar{\gamma}_1^2 X_1 \bar{\psi}_2'(X_1)}{[(\mathrm{i}\bar{n}_1 \bar{X}_1 + 1)\,\bar{\psi}_1(X_1) + \bar{\gamma}_1^2 X_1 \bar{\psi}_1'(X_1)]\,\delta_1 + (\mathrm{i}\bar{n}_1 \bar{X}_1 + 1)\,\bar{\psi}_2(X_1) + \bar{\gamma}_1^2 X_1 \bar{\psi}_2'(X_1)}, \tag{6.49}$$

$$\frac{T}{I} = \left(\frac{\bar{\gamma}_1}{\bar{\gamma}_3}\right)\left(1 + \frac{R}{I}\right)\frac{[\delta_1 \bar{\psi}_1(X_3) + \bar{\psi}_2(X_3)]}{[\delta_1 \bar{\psi}_1(X_1) + \bar{\psi}_2(X_1)]}, \tag{6.50}$$

where
$$\delta_1 = \frac{\bar{\gamma}_3^2 X_3 \psi_2'(X_3) - (\mathrm{i}\bar{n}_3 \bar{X}_3 - 1)\,\psi_2(X_3)}{-\bar{\gamma}_3^2 X_3 \psi_1'(X_1) + (\mathrm{i}\bar{n}_3 \bar{X}_3 - 1)\,\psi_2(X_1)}, \tag{6.51}$$

and the accent denotes differentiation with respect to X.

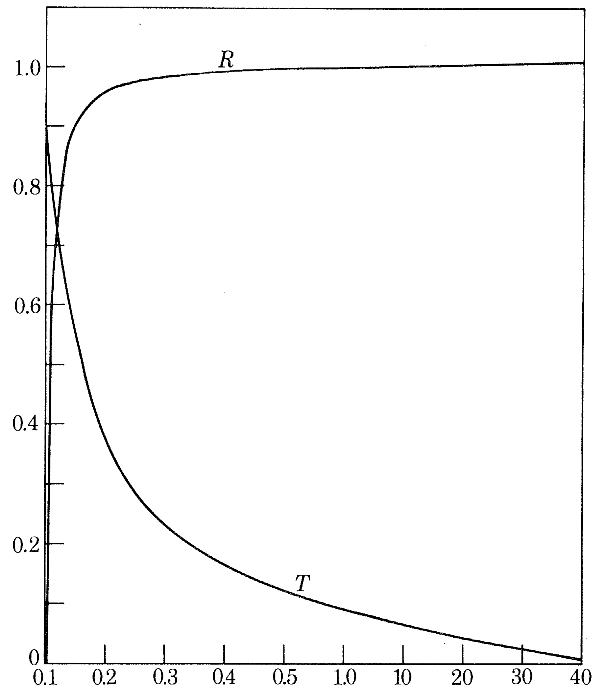

FIGURE 18. The reflexion (R) and transmission (T) coefficients versus $\bar{X}_3$ for case (ii) of § 6.2 drawn for $\bar{\beta} = 0.5$, $\delta = 0.01$, $\bar{X}_1 = -0.1$, $\chi = 0.1$. The curves are little changed by variations in $\bar{\beta}$, χ or $\bar{X}_1$.

Computations of the normal modes equation

$$[(\mathrm{i}\bar{n}_1 \bar{X}_1 + 1)\,\psi_1(X_1) + \bar{\gamma}_1^2 X_1 \psi_1'(X_1)]\,\delta_1 + (\mathrm{i}\bar{n}_1 \bar{X}_1 + 1)\,\psi_2(X_1) + \bar{\gamma}_1^2 X_1 \psi_2'(X_1) = 0, \tag{6.52}$$

for the range of X relevant to this case showed that the system is *stable* for all linear disturbances of the normal mode type. Evaluation of the reflexion coefficient (6.49) showed that over-reflexion is in general possible if m is real but it is extremely weak. Indeed the highest possible value of $|R/I|$ exceeded unity only by a few per cent (see figure 18). Furthermore $|R/I|$ is exactly unity if n_3 is imaginary although $|T/I|$ does not vanish. This can be deduced directly from (6.49)–(6.51). Since the transmitted wave is evanescent we let

$$n_3 = \mathrm{i}n \quad (n > 0), \tag{6.53}$$

the choice of the sign of n being dictated by the radiation condition. Thus δ_1 is real and therefore $|R/I|$ takes the form

$$\frac{R}{I} = \frac{A+\mathrm{i}B}{-A+\mathrm{i}B}, \tag{6.54}$$

where

$$\left.\begin{aligned} A &= \delta_1[\overline{\psi}_1(X_1)+\overline{\gamma}_1^2 X_1 \overline{\psi}_1'(X_1)] + \overline{\psi}_2(X_1) + \overline{\gamma}_1^2 X_1 \overline{\psi}_2'(X_1),\\ B &= \bar{n}_1 \overline{X}_1[\delta_1 \overline{\psi}_1(X_1) + \overline{\psi}_2(X_1)]. \end{aligned}\right\} \tag{6.55}$$

Thus

$$|R/I| = 1, \tag{6.56}$$

and

$$\left|\frac{T}{I}\right|^2 = \left|\frac{\overline{\gamma}_1^2}{\overline{\gamma}_3^2}\right| \frac{4B^2}{(A^2+B^2)} \left[\frac{\delta_1 \overline{\psi}_1(X_3) + \overline{\psi}_2(X_3)}{\delta_1 \overline{\psi}_1(X_1) + \overline{\psi}_2(X_2)}\right]^2 \neq 0. \tag{6.57}$$

It may be noted here that this case corresponds to the situation in which resonant over-reflexion was predicted for smoothly varying shear flows for layers of arbitrary thickness, and the fact that the present system is stable (and over-reflexion is practically non-existent) is mainly due to partial reflexion at the lower 'knee' of the flow profile (at $y = 0$), which reduces the intensity of the wave (particularly for the relatively high frequencies of this case) before it reaches the shear within the layer.

(iii) $|X| \approx 1$

When m is real and $|X|$ takes values on either side of unity, the problem (6.21) in the limit $L \to 0$ is, to leading order, the same as that of the classical critical level for gravity waves *provided* we define the Richardson number by $\frac{1}{4}+\mu^2$. Without going into details the results obtained by Eltayeb & McKenzie (1975) for Richardson numbers slightly greater than $\frac{1}{4}$ hold good here. Thus over-reflexion is not possible and the system is stable. Moreover, absorption of wave energy takes place at the g.w.c.l. even in the limit of vanishing thickness. It can then be deduced that the gravity wave critical latitude in a rotating fluid (occurring where the intrinsic frequency $\hat{\omega}$ matches the Brunt–Väisällä frequency N) is different, at least for thin shear layers, from that present in non-rotating fluids (which occurs where the intrinsic frequency vanishes). It may also be pointed out that, in contrast to the findings in M.M. (see p. 140), the g.w.c.l. is also different from the R.w.c.l. discussed above.

6.3. *Smooth shear profiles*

The example of the linear shear discussed in the preceding subsection does not satisfy the conditions pertaining to the general results obtained in § 6.1 because of the discontinuity in U' at the edges of the shear layer. In order to emphasize the existence of two types of over-reflexion and to stress the importance of the profile of U in effecting either of these two types of over-reflecting régimes we will study the case of a general profile of U in the case when a R.w.c.l. exists within the layer. We will also assume that the layer is so thin that the solution in the neighbourhood of the critical latitude represents a reasonable approximation to the solution in the whole layer. Now assuming that $F_c'' \neq 0$ and noting that $L \ll 1$, we see that

$$\alpha = fF_c''/U_0(F_c')^2 + O(L) \tag{6.58}$$

and hence the region influenced by the R.w.c.l. (which occurs at $X = 0$) is much larger than in the case of the linear shear (where $F_c'' = 0$). The solution within the shear then takes the form (5.6) with y replaced by Y. The associated boundary conditions here demand that both ψ and ψ' are continuous at $Y = 0, 1$. Consequently

$$\frac{R}{I} = \frac{\mathrm{i}(n_1 \pm \pi\alpha) - (\mathrm{i}n_3 + \alpha\gamma)}{\mathrm{i}(n_1 \mp \pi\alpha) + (\mathrm{i}n_3 + \alpha\gamma)}, \quad \frac{T}{I} = 1 + \frac{R}{I}, \tag{6.59}$$

in which $$\gamma = \ln(-1+Y_c^{-1}), \tag{6.60}$$

with the subscript c denoting values at the R.w.c.l., measures the position of the critical latitude relative to the centre of the layer. Now the assumption that the layer is very thin together with the hypothesis that n_1 is real demand that the present case is relevant only to the case of a westerly decreasing with latitude or an easterly increasing with latitude. In either case n_3 is imaginary and has the form (6.53). Thus

$$\left|\frac{R}{I}\right|^2 = 1 \pm \frac{2\pi n_1 \alpha}{(n_1 \mp \pi\alpha)^2 + (n-\gamma\alpha)^2}, \tag{6.61}$$

and over-reflexion will take place if

$$\alpha n_1 \gtrless 0 \quad \text{for} \quad kU_c' \gtrless 0. \tag{6.62}$$

Remembering that $U_0 = U_3 - U_1$ and $\operatorname{sgn}(U_c') = \operatorname{sgn}(U_0 F_c')$ and realizing that $kn_1 > 0$ for the present case, we can write the condition for the occurrence of over-reflexion as

$$\operatorname{sgn}(fF_c' F_c'') > 0. \tag{6.63}$$

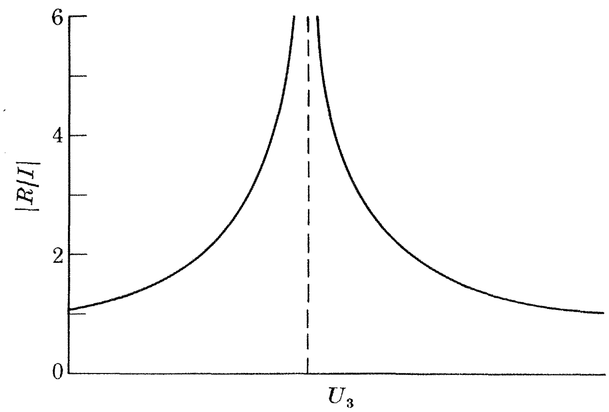

FIGURE 19. A schematic representation of the over-reflexion of § 6.3.

Thus over-reflexion can occur if $F_c'' > 0$ for the westerly and if $F_c'' < 0$ for the easterly, assuming $f > 0$. Moreover, an analysis of the normal modes equation

$$\mathrm{i}(n_1 \mp \pi\alpha) + \alpha\gamma - n = 0 \tag{6.64}$$

shows that the only neutral mode of stability is given by

$$c = \frac{\omega}{k} = \frac{\alpha^2 U_1 + \gamma^2 U_3}{\alpha^2+\gamma^2} - \frac{(U_3-U_1)\,k^2}{\pi^2(\alpha^2+\gamma^2)}(1+\chi) \tag{6.65}$$

and $|R/I|$ is infinite when (6.65) is satisfied.

Now $$|R/I|^2 = 1 + 2\pi\,|\alpha n_1|\operatorname{sgn}(fF_c' F_c'')/\{(n_1 \mp \pi\alpha)^2 + (n-\gamma\alpha)^2\} \tag{6.66}$$

and for given k, ω, χ, U_1 and F over-reflexion takes place even if (6.65) is not obeyed provided U_3 is not too far from the value, $\bar{U}$, satisfying (6.65) and as U_3 increases to $\bar{U}$ the reflexion coefficient becomes infinite and resonance occurs. This is confirmed by the expression

$$\left|\frac{T}{I}\right|^2 = \frac{4n_1^2}{(n_1 \pm \pi\alpha)^2 + (n-\alpha\gamma)^2}, \tag{6.67}$$

for the transmission coefficient which also tends to infinity as U_3 takes the value $\bar{U}$. This situation is represented schematically in figure 19 to facilitate comparison with the other type of over-reflexion discussed in § 6.2 above.

7. Conclusions

The study of the local dispersion relation (3.2) for subacoustic flow speeds showed that propagation of gravity waves on a beta-plane is possible in the presence of latitudinally sheared zonal flows only if the waves propagate vertically, i.e. only if m is real. If these waves are evanescent in the vertical direction they will also be evanescent in the latitudinal direction and therefore they will, if excited, propagate eastward in a thin region surrounding the latitude on which the source is situated. In this way they will not be efficient in transferring energy and momentum in latitudinal or vertical directions. However, if they are propagating vertically then they can also propagate latitudinally and can transfer energy and momentum in both directions (see § 4). Moreover, their critical latitudes are very different from the classical critical levels of Booker & Bretherton (1967). Although both total wave energy flux and momentum transferred in the northward direction experience a negative jump across each critical latitude, the attenuation factors for the two waves there (one northward-going and the other southward-going) is *not* the same if the fluid is *non*-Boussinesq because the waves exhibit valve-like behaviour there which allows one of the waves to cross the critical latitude unattenuated. The valve behaviour at these critical latitudes is found to be responsible for the escape of gravity waves from regions of high shear flow to those of low shear flow and thereby facilitate the 'coupling' between high and low frequency waves on a beta-plane.

In § 6 the reflexion and transmission of Rossby-gravity waves by a finite latitudinal shear has been investigated. It is shown that the presence of a Rossby wave critical latitude (occurring where $\hat{\omega} = 0$) has a profound effect on the reflectivity, transmissivity and stability of planetary waves in the presence of a finite shear layer. The influence of this type of critical latitude on the intensity of the wave is found to be measured by the parameter α which is equal to the product of the ratio of the planetary vorticity to the basic flow vorticity and the logarithmic derivative of the potential vorticity at the critical latitude. All three properties are found to be strongly influenced by the magnitude and sign of α. In the linear shear the potential vorticity gradient is the same as the planetary vorticity gradient and $\alpha < 0$ for $f > 0$. Here over-reflexion is possible but instability is absent. However, in the case of nonlinear shear (and because the shear is thin) the gradient of potential vorticity has the same magnitude (but different sign) as the gradient of the shear flow vorticity and the sign of α depends on the sign of the potential vorticity (as measured by the curvature of the flow F''_c). Over-reflexion depends on the sign of the product of the flow vorticity, the planetary vorticity and the potential vorticity gradient as measured on the critical latitude (cf. (6.63). If the product is positive over-reflexion takes place but the system is unstable, and in contrast to the linear shear, over-reflexion here is resonant.

The influence of the gravity wave critical latitude on a beta-plane on the reflectivity and stability of the shear is found to be different from that of the classical gravity wave critical level. The g.w.c.l. is always associated with wave energy absorbtion even in the limit of vanishing thickness. The reason for this can be traced down to the fact that the equivalent Richardson number approaches $\frac{1}{4}$ as the thickness of the layer tends to vanish. In this respect the g.w.c.l. is similar to the critical level of hydromagnetic-gravity waves (Eltayeb 1980).

The reflectivity and transmissivity properties of thin shear layers have generally been believed to be adequately represented by the corresponding vortex sheet. The above analysis, however, shows marked disagreement between the two cases and it is of interest to examine the reasons for the wide disagreement located here, bearing in mind the close agreement obtained by Eltayeb & McKenzie (1975) for gravity waves incident on a linear shear in a Boussinesq non-rotating fluid. After studying a number of other situations (see Eltayeb 1977, 1980; El Sawi & Eltayeb 1978) it seems that whether the thin shear and the corresponding vortex sheet will yield the same results or not depends on the equivalent Richardson number, R_E, of the system and on the flow profile. R_E is α for R.w.c.l., $\frac{1}{4}+\mu^2$ for the g.w.c.l. and Ri for the classical gravity wave critical level. The significance of R_E is that it measures the jump in the intensity of the wave (as measured by $\mathscr{A}$) across the singularity. If R_E approaches zero in the limit $L \to 0$, as it does for the case $R_E = Ri$ then the influence of the critical latitude (or level) is negligible in this limit and the thin shear and vortex (or current-vortex) sheet will exhibit similar results. However, if R_E tends to a finite non-zero value as L approaches zero the shear and sheet can be expected to lead to different reflectivity and stability properties. It is also found that discontinuities in the flow at the edges of a shear layer have the effect of reducing the interaction between wave and shear. The case of discontinuous shear (see § 6.2) tends to yield results which are, to leading order, similar to those of the vortex sheet. However, the smooth shear is associated with different reflectivity and stability properties (see § 6.3 above and Eltayeb 1980). It may then be concluded that only those shear layers of the linear type for which the equivalent Richardson number tends to zero as the thickness of the shear approaches zero may be expected to exhibit results similar to the vortex and/or current sheet.

The search for over-reflecting régimes in the absence of critical latitudes within the shear layer revealed that such régimes are present if the vertical wavenumber, m, is real (and gravity waves are present). Here over-reflexion occurs provided the incident and transmitted waves are of different types, i.e. one is a Rossby wave (slightly modified by gravity) and the other is a gravity wave (slightly modified by rotation). However, such régimes are found to be associated with the presence of instability and hence over-reflexion here is resonant. By taking into account the various situations in which over-reflexion is predicted in the absence of critfcal levels (Eltayeb 1977; El Sawi & Eltayeb 1978, § 6 above) it may be conjectured that a pre-requisite condition for the existence of over-reflexion in the absence of critical levels (or latitudes) is the existence of hybrid wave-motions, i.e. systems in which at least two types of wave motions, which can exist independently, are coupled together.

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